Abstract

We consider the geometry of planes, spheres and surfaces in the context of classical differential geometry. Geometric concepts which generalize from planes to spheres to surfaces are emphasized. This includes straight lines in planes which become geodesics on spheres and surfaces. Triangles in planes which become geodesic triangles on spheres and surfaces. The properties of such triangles are sensitive to the curvature of the sphere and general curved surfaces. Differentiation of functions and vectors on a plane generalize to covariant differentiation on spheres and curved surfaces. Covariant differentiation lead to constructions of quantities which measure curvature locally on spheres and surfaces. The spatial distribution of geodesics is controlled by differential equations which are sensitive to the curvature of the surfaces they are embedded in. The equations of Jacobi fields, the two dimensional analog of geodesic deviation, are obtained and studied.

This lecture supplements material in the textbook: Special Relativity, Electrodynamics and General Relativity: From Newton to Einstein (ISBN: 978-0-12-813720-8). The term “textbook” in these Supplemental Lectures will refer to that work.

Keywords: Differential Geometry, curvature, geodesic, covariant differentiation, Gaussian curvature, holonomy, Jacobi fields.
Planar Geometry: Setting the Stage.

Let’s begin on familiar ground: two dimensional Euclidean space $\mathbb{R}^2$. We can take $\mathbb{R}^2$ to be the $x$-$y$ plane with the $z$-direction normal to any point therein. We have all done geometry in $\mathbb{R}^2$ and know about straight lines, angles, triangles, open and closed curves, two dimensional integrals, etc.

The first thing to ask about Euclidean space is: What are its symmetries? In the case of $\mathbb{R}^2$ it is clear that rigid translations and rotations are symmetries: every region in $\mathbb{R}^2$ is equivalent to every other. Translations and rotations are isometries of $\mathbb{R}^2$: they preserve relative distances between points and therefore they preserve angles between intersecting curves.

Consider a curve $\vec{r}(t)$ in $\mathbb{R}^2$. Some examples are shown in Fig. 1. There is a straight line in Fig. 1a, a curved line in Fig. 1b, a simple, smooth, closed curve in Fig. 1c and a curve with three smooth sections and three vertices in Fig. 1d.
Curves in R² were discussed in the textbook and in supplementary lecture #8. Let’s recall some facts, properties and derivations. Let’s use the notation (x, y) for points on the plane in orthonormal coordinates. The unit vectors along the x and y axes will be denoted \( \hat{i} \) and \( \hat{j} \) as usual. They are orthonormal so \( \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = 1 \) and \( \hat{i} \cdot \hat{j} = 0 \). A circle in this coordinate system would be written \( \vec{r}(t) = (a \cos t, a \sin t) \) where \( a \) is the radius of the circle and \( t \) is a parameter on the curve. The range \( 0 < t < 2\pi \) sweeps out the entire circle as shown in Fig. 2,
A convenient parameter to label a curve is its arclength,

\[ s(t) = \int_0^t |\vec{r}'(t)| \, dt \quad 1. \]

where \( \vec{r}'(t) \equiv d\vec{r}/dt \) and \( |\vec{r}'(t)| = (x'(t)^2 + y'(t)^2)^{1/2} \). It follows from Eq. 1 that \( ds/dt = |\vec{r}''(t)| \). If we think of \( t \) as “time”, then \( ds/dt \) is the speed of a particle traveling on \( \vec{r}(t) \). In differential geometry it proves convenient to choose \( t \) itself to be the arclength. In that case we write \( \vec{r}(s) \) and note that \( |\vec{r}'(s)| = 1 \): the curve has “unit speed”. Why is this convenient? Because in this case \( \vec{r}'(s) \cdot \vec{r}'(s) = 1 \) implies, by differentiation, \( \vec{r}'''(s) \cdot \vec{r}'(s) = 0 \) and the “acceleration” is perpendicular to the “velocity”, as depicted in Fig. 3,
In physics applications the acceleration of a particle can increase the linear speed or it can increase the curvature, change of direction, of the curve. So, writing \( \vec{r}(s) \) focuses the formalism on the curvature which is of central interest here. In fact, the magnitude \(|\vec{r}''(s)| = \kappa(s)\) is called the curvature of \( \vec{r}(s) \) at \( s \). We have studied \( \kappa(s) \) extensively in the textbook and in other supplementary lectures, especially in #8, and will review that material below. The first observation to make is that the vanishing of \( \kappa(s) \) picks out straight lines in \( \mathbb{R}^2 \). If \( \vec{r}(s) \) is a straight line, then \( \vec{r}(s) = \hat{u}s + \hat{v} \) where \( \hat{u} \) and \( \hat{v} \) are fixed unit vectors. Then we calculate \(|\vec{r}''(s)| = \kappa(s) = 0\). Conversely, if \( \kappa(s) = |\vec{r}''(s)| = 0\), then by integration \( \vec{r}(s) = \hat{u}s + \hat{v} \) and the curve is a straight line.

It is convenient to introduce the unit vector \( \vec{t}(s) = \vec{r}'(s) \) which is tangent to the curve \( \vec{r}(s) \) and points in the direction of increasing \( s \). Then we write \( \vec{r}''(s) = \kappa(s)\vec{n}(s) \) where \( \vec{n}(s) \) is a unit vector perpendicular to \( \vec{t}(s) \) which points in the direction of the acceleration as illustrated in Fig. 3. For planar curves one can also assign a sign to the curvature \( \kappa(s) \). Let \( \{\hat{i}, \hat{j}\} \) be the natural
basis for $R^2$, constant unit vectors along the $x$ and $y$ axes. Then choose the basis $\{\vec{t}(s), \vec{n}(s)\}$ to have the same orientation as $\{\hat{i}, \hat{j}\}$. Then define the curvature through the identity $\vec{t}'(s) = \kappa(s)\vec{n}(s)$ and $\kappa(s)$ may be positive or negative. The reader should consult supplementary lecture #8 for illustrations and applications. Three dimensional curves and the characterization of such curves through their “torsion” $\tau(s)$ and a third unit vector, the “bi-normal”, are also discussed there. The only point we want to make here is that the curvature $\kappa(s)$ indicates how quickly the curve $\vec{r}(s)$ deviates from a straight line. To see this, make a Taylor series expansion of $\vec{r}(s)$ about $s = 0$,

$$\vec{r}(s) = \vec{r}(0) + s\vec{t}'(0) + \frac{1}{2}s^2\vec{t}''(0) + \frac{1}{6}s^3\vec{t}'''(0) + \cdots$$

where the neglected terms are $O(s^4)$ and higher. Eq. 2 can be simplified because $\vec{r}'(0) = \vec{t}(0) \equiv \hat{t}$ and $\vec{t}''(0) = \kappa(0)\vec{n}(0) \equiv \kappa\vec{n}$ where we leave the argument $s = 0$ off quantities where the situation is clear. We need the quantity $\vec{t}'''(0)$ in Eq. 2,

$$\vec{t}'''(0) = (\kappa\vec{n})' = \kappa'\vec{n} + \kappa\vec{n}'$$

The rate of change of the normal vector $\vec{n}(s)$ follows from the fact that $\vec{n}(s) = \hat{k} \times \vec{t}(s)$ where $\hat{k}$ is the unit normal in the $z$ direction which is normal to the $x$-$y$ plane. So,

$$\vec{n}'(s) = \hat{k} \times \vec{t}'(s) = \kappa(s)\hat{k} \times \vec{n}(s) = -\kappa(s)\vec{t}(s)$$

Now Eq. 2 becomes,

$$\vec{r}(s) - \vec{r}(0) = \left(s - \frac{1}{6}\kappa^2s^3\right)\vec{t} + \left(\frac{1}{2}\kappa s^2 + \frac{1}{6}\kappa' s^3\right)\vec{n} + \cdots$$

This expression is clearer if we translate and rotate the $x$-$y$-$z$ coordinate system so that its origin coincides with $s = 0$, $\vec{t}$ lies in the $x$ direction and $\vec{n}$ lies in the $y$-direction. Then we have $\vec{r}(s) - \vec{r}(0) = (x(s), y(s))$ with,

$$x(s) = s - \frac{1}{6}\kappa^2s^3 + \cdots, \quad y(s) = \frac{1}{2}\kappa s^2 + \frac{1}{6}\kappa' s^3 + \cdots$$

which is plotted in Fig. 4,
The figure shows that $\kappa$ indicates how quickly the curve turns from its tangent in quantitative terms. It will be interesting to see how this idea generalizes to curved surfaces in $R^3$. It will turn out that the second fundamental form of the surface plays the analogous role, how quickly does a curved surface deviates from its tangent plane.

It addition to the local properties of geometric objects, we shall be interested in global properties as well. For example, we might consider a closed, simple (non-intersecting), regular curve in $R^2$ as shown in Fig. 5,
As $s$ increases from $s = 0$ and we trace the curve around one turn back to $s = 0$ through an arclength $l$, $\mathbf{t}(s)$ rotates once through $2\pi$ radians in the counter-clockwise direction. Since $\mathbf{t}(s)$ is a unit vector, we can parametrize it with its angle off the $x$ axis. Call the angle $\theta(s)$ and write,

$$\mathbf{t}(s) = (\cos \theta(s), \sin \theta(s))$$

Then $\mathbf{t}'(s) = \theta'(s)(-\sin \theta(s), \cos \theta(s)) = \theta'(s)\mathbf{n}(s)$ and we can identify the curvature,

$$\kappa(s) = d\theta/ds$$

For the curve shown in Fig. 5,

$$\int_0^l \kappa(s)ds = \theta(l) - \theta(0) = 2\pi$$

We will generalize Eq. 3, 4 and 5 to curves embedded in curved surfaces in $\mathbb{R}^3$ in the later sections of this lecture.
One point we learn from these exercises is that straight lines are picked out by the criterion \( \kappa(s) = 0 \) for all \( s \). Straight line are the shortest paths between any two points in the plane. If we consider a point \( P \) and a direction \( \vec{v} \), then it is clear that there is a unique straight line through \( P \) in the direction \( \vec{v} \). The generalization of this trivial observation to curved surfaces will be non-trivial. We will introduce the generalization – geodesics – later in this lecture. This material will supplement and support the discussions of geodesics in the textbook.

Straight lines are the building blocks of closed regions in \( R^2 \). The simplest are triangles. We all learn in school that the sum of the interior angles of a triangle is precisely \( \pi \) in all cases. This can be proved by considering the tangent vector to the straight lines making up the triangle as they are traversed. At vertex \( i \) where \( \vec{t} \) rotates by \( \theta_i \), the interior angle is \( \varphi_i = \pi - \theta_i \) as shown in Fig. 6,

![Fig. 6 The turning angle traversing a triangle](image)

Since \( \vec{t}(s) \) turns through \( 2\pi \) radians as the triangle is traversed,
\[ \theta_1 + \theta_2 + \theta_3 = 2\pi \]

In terms of \(\varphi_i\),

\[ (\pi - \varphi_1) + (\pi - \varphi_2) + (\pi - \varphi_3) = 2\pi \]

So,

\[ \varphi_1 + \varphi_2 + \varphi_3 = \pi \quad 6. \]

And we learn the very familiar fact that the sum of the interior angles of any triangle is \(\pi\). The generalization of Eq. 6 to spheres and general curved surfaces will teach us a great deal about differential geometry and lead us to the Gauss-Bonnet Theorem, the most influential result in the subject. Along the way we will have to understand geodesics on curved surfaces, geodesic triangles and the nature of the geodesic curvature of curves embedded on surfaces and the Gaussian curvature of surfaces.

It is obvious that the relation \(\varphi_1 + \varphi_2 + \varphi_3 = \pi\) has nothing to do with the size of the triangle under consideration. In fact, in the context of \(R^2\) what does size mean? \(R^2\) contains no inherent scale. In Euclidean geometry triangles which are similar (have the same interior angles) are not necessarily congruent. A length relation between two triangles must be specified to establish congruence. The situation for curved surfaces is different because they come with an intrinsic scale, their Gaussian curvature \(K\). In the case of a sphere of radius \(R\) where \(K = R^{-2}\), similar geodesic triangles are necessarily congruent! Some of these points were made in the supplementary lecture #4 on planar, spherical and hyperbolic geometries and will be developed further below.

A central postulate in Euclidean geometry is the Parallel Postulate. One form of the Postulate states that given a straight line \(L_1\) and a point \(P\) outside that line, there is only one straight line \(L_2\) through \(P\) that can be extended indefinitely in both directions and \(L_2\) never intersects \(L_1\). This statement of the Parallel Postulate is equivalent to others the reader may have learned about but it must be understood within the context of the other (four) postulates of Euclidean geometry. As we already discussed in supplementary lecture #8, the projective plane, basically the geometry of the surface of a sphere \(S^2\) with anti-podal points identified, is an example of a space with positive Gaussian curvature where the Parallel Postulate is not valid. Similarly, hyperbolic
geometry, based on an abstract space of negative constant Gaussian curvature, also violates the fifth postulate of Euclidean geometry. The Poincare Disk and Upper Half Complex Plane models were discussed in detail in supplementary lecture #4. There we also saw that geodesic triangles on \(S^2\) have \(\sum \varphi_i > \pi\) and geodesic triangles on the Poincare Disk have \(\sum \varphi_i < \pi\). The deficit \((\sum \varphi_i - \pi)\) is, in fact, controlled by the Gaussian curvature of the space and is predicted quantitatively by the Gauss-Bonnet Theorem. More later.

Another related property is the equation for the distance between two geodesics that start from a common point with an angle \(\varphi\) between them. On a plane the two straight lines diverge proportional to the length of each geodesic as is clear in Fig. 7.

![Fig. 7 Spreading of straight lines on a plane](image)

We shall see that on a surface with positive Gaussian curvature \(K > 0\) the distance between the geodesics, the “geodesic deviation”, grows at a slower rate. And on a surface with negative \(K\), the geodesic deviation grows at a faster rate. We will devote considerable effort below to make these points clear. Geodesic deviation lies at the heart of curvature in the context of differential
geometry and is a particularly important observable in general relativity. The subject of Jacobi fields is the relevant topic in classical differential geometry.

**Spherical Geometry**

Consider the geometry on the surface of a sphere $S^2$. We can locate any point $P$ on the sphere using the angles $\theta$ and $\phi$ shown in Fig. 8,

![Fig. 8 Spherical coordinates](image)

We read off the figure, in Cartesian coordinates,

$$\vec{r}(\theta, \phi) = R(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$
where $R$ is the radius of the sphere. At the point P there is a unit normal $N(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$. The derivatives $\tilde{r}_\theta = \partial \tilde{r} / \partial \theta$ and $\tilde{r}_\varphi = \partial \tilde{r} / \partial \varphi$ and the normal $\tilde{N}(\theta, \varphi)$ comprise an orthogonal triad at the point P,

$$
\tilde{r}_\theta = R(\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta)
$$

$$
\tilde{r}_\varphi = R(-\sin \theta \sin \varphi, \sin \theta \cos \varphi, 0)
$$

Therefore, $\{ \tilde{r}_\theta, \tilde{r}_\varphi \}$ spans the tangent plane at point P on the sphere.

Our chore is to find the geodesics on the sphere. Take the north pole as the initial position of a geodesic and consider a geodesic in the direction $\varphi = \varphi_0$. Consider the path $\tilde{r}(\theta, \varphi_0).$ It has the tangent $\tilde{r}_\theta(\theta, \varphi_0).$ Clearly $\tilde{r} \cdot \tilde{r}_\theta = 0.$ Since $\tilde{r} = RN$ and $\tilde{N} \cdot \tilde{N} = 1,$ it follows that $\tilde{r}_\theta \cdot \tilde{r} = \tilde{r}_\varphi \cdot \tilde{r} = 0.$ In addition, from Eq. 8 we can calculate the “acceleration”,

$$
\tilde{r}_{\theta\theta} = -R(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) = -R \tilde{N}
$$

So, $\tilde{r}_{\theta\theta} \cdot \tilde{r}_\theta = \tilde{r}_{\theta\theta} \cdot \tilde{r}_\varphi = 0,$ and the acceleration has no projection onto the tangent plane. This curve has no curvature within the tangent plane, so a two dimensional creature living on $S^2$ concludes that each path $\tilde{r}(\theta, \varphi_0)$ having fixed $\varphi = \varphi_0$ is “straight”. In the textbook and supplementary lecture #8 we would conclude that this path has vanishing geodesic curvature $\kappa_g$ and is a geodesic. These paths, which are the intersection of planes that pass through the origin and the sphere are “Great Circles”, famous in navigation. We see that there is a geodesic, or great circle, in every direction $\varphi_0$ starting from the north pole. This fact must be true of every point on $S^2$ by symmetry. The sphere shares this property with the plane – there is a geodesic in every direction starting at any point P. Later we shall verify this property for any curved surface although the shape of each geodesic will be different in each case.

These points should be familiar to the reader from the textbook and supplementary lecture #8. We should also point out some properties of the geodesics globally on the sphere. It is clear that different great circles intersect each other twice as they circumnavigate the globe. Great circles which begin at the north pole and diverge with an angle $\Delta \varphi$ between them, become parallel at the equator where all the $\tilde{r}_\theta$ point in the $-z$ direction and then they intersect at the south pole. One says that the south pole is a “conjugate” point for these geodesics. We will study
the way geodesics fan apart from their starting point and coalesce toward their conjugate point below. The controlling parameter in this analysis will be the Gaussian curvature $K$.

Now that we know the geodesics on the sphere we can construct “triangles” from them and study the effect of the sphere’s curvature. This discussion is a precursor to the Gauss-Bonnet Theorem and can be done with elementary geometric considerations because of the simplicity and symmetries of $S^2$. Begin by considering Fig. 9 which shows three great circles forming a geodesic triangle with interior angles $\alpha, \beta, \gamma$.

![Fig. 9 Geodesic triangle on a sphere](image)

We want the area of the spherical triangle $\Delta$ which has interior angles $\alpha, \beta, \gamma$. We begin with the area of a “double lune”, the area between two great circles,
Note that there is a similar figure in the back of the unit sphere in Fig. 10. Since the total area of the unit sphere is $4\pi$, the area of the double lune is $4\phi$ with $\phi$ restricted to the range $0 < \phi < \pi$.

Now let’s count the double lunes in Fig. 9. There are double lunes with angles $\alpha, \beta, \gamma$ and they cover the sphere with some over counting: $\Delta$ and its image on the far side of the sphere, call it $\Delta'$, are in all three double lunes. All other points on $S^2$ are in only one of the double lunes. So, the total surface area of $S^2$ is,

$$4\pi = 4(\alpha + \beta + \gamma) - 2\Delta - 2\Delta'$$

10.

where $\Delta$ indicates the area of the triangle. Clearly $\Delta'$ is a copy of $\Delta$, so they share the same (surface) areas and $\Delta' = \Delta$. Solving Eq.10,

$$\Delta = (\alpha + \beta + \gamma) - \pi$$

11.

Let’s replace the unit radius of $S^2$ with a general length $R$, so Eq. 11 becomes,

$$\Delta = ((\alpha + \beta + \gamma) - \pi)R^2$$

12.a
As we know from the textbook, the Gaussian curvature of a sphere of radius \( R \) is \( K = R^{-2} \), so Eq. 12a can be written in the form of the Gauss-Bonnet Theorem as

\[
K\Delta = (\alpha + \beta + \gamma) - \pi \tag{12.b}
\]

We learn several things from this result. First, the sum of the interior angles of a spherical triangle is generally greater than \( \pi \) and the positive deficit determines the area of the triangle for a given sphere. Unlike triangles in \( R^2 \) which for a given fixed \( \alpha, \beta, \gamma \), can come in arbitrary sizes, on the sphere the angular deficit determines the size of the triangle. The Gaussian curvature \( K \) provides the scale to make such a relation possible. For large spherical triangles, the angular deficit becomes large. For example, the spherical triangle in Fig. 11 can have an angular deficit reaching \( 2\pi \),

![Fig. 11 Angular deficit of a spherical triangle](image)

In this case the great circle \( BC \) is on the equator and for a sphere with unit radius, \( \Delta \) can be as large as a half of the area of the sphere, \( 4\pi/2 = 2\pi \), which is just the maximum value of \( \alpha \).
The other extreme of small triangles is also informative. In that case, for triangles $\Delta$ whose sizes are small on the scale of the curvature, $K\Delta \to 0$, we see that the angular deficit vanishes, $(\alpha + \beta + \gamma) \to \pi$. This is just reminding us that the short distance geometry of the curved surface becomes Euclidean! The tangent planes at any point on the sphere approximate the lengths and angles of curves on the surface locally. Although we can measure the Gaussian curvature $K$ of a surface locally, we have to expose higher order corrections to Euclidean geometry to do so. This is a fundamental property of classical differential geometry as well as Riemannian geometry.

We can understand the geometry of spherical triangles in more detail. Consider the spherical triangle in Fig. 12,

![Fig. 12 Angles and sides of a spherical triangle](image)

Note that the arclengths $a$, $b$ and $c$ are the same as the interior angles $a$, $b$ and $c$ at the origin $O$ of the sphere. An exercise in vector analysis [4] in $\mathbb{R}^3$ gives the relation between the sides of the spherical triangle $a$, $b$ and $c$ and the interior angles $\alpha, \beta, \gamma$ of the spherical triangle,
\[
\sin a \sin b \cos \gamma = \cos c - \cos a \cos b \tag{13.a}
\]

and analogous relations for \(\cos \alpha\) and \(\cos \beta\). The reader should recognize this result as the spherical analogue of the “law of cosines” formula for planar triangles. Consider the planar triangle in Fig. 13,

![Fig. 13 Law of cosines for a planar triangular](image)

Adding the vectors in Fig. 13, \(\vec{c} = \vec{b} - \vec{a}\), we calculate \(\vec{c}^2 = \vec{b}^2 + \vec{a}^2 - 2|\vec{a}||\vec{b}| \cos \gamma\) which allows us to calculate the angle between two sides of the triangle in terms of the lengths of each side. The reader should check that Eq. 13a reduces to the “law of cosines” for small spherical triangles.

From the perspective of this lecture a more interesting identity follows from the “dual” of the triangle [4] in Fig. 13. The dual triangle \(\Delta A'B'C'\) is constructed from \(\Delta ABC\) by “interchanging” angles and lengths \(a \rightarrow \pi - \alpha\), \(b \rightarrow \pi - \beta\), \(c \rightarrow \pi - \gamma\) and \(\alpha \rightarrow \pi - a\), \(\beta \rightarrow \pi - b\) and \(\gamma \rightarrow \pi - c\). Then the spherical triangle formula Eq. 13a becomes,
\[ \sin \alpha \sin \beta \cos c = \cos \gamma - \cos \alpha \cos \beta \]

and analogous relations for \( \cos a \) and \( \cos b \). This relation shows that we can calculate the lengths of the sides of the triangle \( ABC \) just in terms of its interior angles. In other words, in spherical geometry, triangles which are similar (have the same interior angles) are congruent (have the same sides, lengths)! The Gaussian curvature indeed sets the scale of lengths in curved geometries. The reader should consult the references [1-4] for more details in spherical geometry.

Now let's consider the geodesics, great circles of the sphere again and consider their distribution near the north pole. We are interested in comparing the rate at which they spread out along their lengths compared to the same problem on a plane. The geometry of the situation is shown in Fig. 14,

![Fig. 14 Spreading of geodesics on a sphere](image)

We see two great circles starting at the north pole with an angle \( \varphi \) between them. After an arclength \( s \) has been traversed, the two great circles are a distance \( \epsilon \) apart as measured on the coordinate circle of fixed polar angle \( \theta \). We read off the figure that,

\[ \epsilon = \varphi \mathcal{R} \sin\left(\frac{s}{\mathcal{R}}\right) \]
We see that the great circles initially diverge, reach a maximum distance apart on the equator where their tangents are parallel, and then intersect at the south pole, called a “conjugate point” because of this fact. If we continue the great circles beyond the south pole, they are no longer the shortest distances between the initial and final points on the arc – the shorter path is provided by the great circle in the same plane but on the back side of the axis of the sphere. This observation indicates that geodesics are only “shortest paths” on curved surfaces if they are “short enough”!

Other examples on other surfaces such as cylinders are easily pointed out [1-4].

We see that the shortest distance between the geodesics satisfies harmonic dependence on the arclength. It satisfies the harmonic oscillator local differential equation,

$$ \frac{d^2 \epsilon}{ds^2} = -K \epsilon $$

where $K$ is the Gaussian curvature, $K = 1/R^2$. It is Eq. 15 that will guide us to the generalization of these considerations to curved surfaces, and not the global expression Eq. 14. In the generalization $K$ will be replaced by the spatially dependent Gaussian curvature $K(\theta, \varphi)$ and $\epsilon$ will become the geodesic deviation vector as discussed in the textbook. In planar geometries $K = 0$ and $\epsilon$ grows linearly with $s$, as we pointed out earlier, and in hyperbolic geometries where $K < 0$, $\epsilon$ grows faster than linearly for small $s$. We have seen this behavior in our discussion of hyperbolic geometry of the Poincare Disk in supplementary lecture #4. The reader might also consult Fig. 12.5 in the textbook.

It should be clear from this example that the generalization of plane polar coordinates to surfaces will be very handy. These generalizations are called Gaussian polar coordinates. In the plane we read off Fig. 15.
Here $\rho$ is the length of $\vec{r}(\rho, \phi)$ and the metric reads,

$$d\vec{r}^2 = dx^2 + dy^2 = d\rho^2 + \rho^2 d\phi^2$$

In plane polar coordinates, the coordinate vectors vary in space and are $\vec{r}_\rho = \partial \vec{r} / \partial \rho$ and $\vec{r}_\phi = \partial \vec{r} / \partial \phi$ which are orthogonal everywhere but are not unit vectors. The generalization to the sphere is familiar and has already been recorded in Eq. 7 and 8. The metric written in terms of $\theta$ and $\phi$ reads,

$$d\vec{r}^2 = R^2 d\theta^2 + R^2 \sin^2 \theta d\phi^2$$  \hspace{1cm} 16a$$

For small $\theta = s/R$, in the vicinity of the north pole, $\sin \theta = \theta - \theta^3/6 + \cdots$, so

$$d\vec{r}^2 = R^2 d\theta^2 + s^2 \left( 1 - \frac{s^2}{3R^2} + \cdots \right) d\phi^2$$  \hspace{1cm} 16b$$

To calculate the Gaussian curvature at the north pole, we consider a circle of geodesic radius $s$. One definition of the Gaussian curvature considers the circumference of a geodesic circle of

Fig. 15 Planar polar coordinates
of radius s and compares it to the Euclidean circumference, $2\pi s$. In detail,

$$K = \frac{3}{\pi} \lim_{s \to 0} \left( \frac{2\pi s - C(s)}{s^3} \right)$$  \hspace{1cm} (17)$$

as introduced in the textbook. Here, using Eq. 16b,

$$C(s) = \mathcal{R} \int_0^{2\pi} \sin \left( \frac{s}{\mathcal{R}} \right) d\varphi = 2\pi \mathcal{R} \sin \left( \frac{s}{\mathcal{R}} \right) = 2\pi s \left( 1 - \frac{1}{6} \frac{s^2}{\mathcal{R}^2} \right) = 2\pi s - \frac{2\pi}{6} \frac{s^3}{\mathcal{R}^2} + \cdots$$  \hspace{1cm} (18)$$

Substituting into Eq. 17 we find the expected result, $K = 1/\mathcal{R}^2$.

This exercise suggests that for a general curved surface the metric in polar coordinates should have the form,

$$d\vec{r}^2 = ds^2 + G d\varphi^2$$  \hspace{1cm} (19a)$$

where $G = G(s, \varphi)$ and,

$$\lim_{s \to 0} \sqrt{G} = 0, \quad \lim_{s \to 0} \left( \frac{\sqrt{G}}{s} \right) = 1$$  \hspace{1cm} (19b)$$

Then,

$$K = -\frac{\left( \frac{\sqrt{G}}{s} \right)_{ss}}{\sqrt{G}}$$  \hspace{1cm} (19c)$$

These local statements are suggested by Eq. 16b. In particular, for the sphere $\sqrt{G} = \mathcal{R} \sin(s/\mathcal{R})$, so Eq. 19b is true by inspection. Eq. 19c reads,

$$K = -\frac{\left( \frac{\sqrt{G}}{s} \right)_{ss}}{\sqrt{G}} = \frac{1}{\mathcal{R}} \frac{\sin(s/\mathcal{R})}{\mathcal{R} \sin(s/\mathcal{R})} = \frac{1}{\mathcal{R}^2}$$  \hspace{1cm} (19d)$$

We shall see below that Eq. 19a-c are in fact generally true for Gaussian polar coordinates for a general curved surface where $K(s, \varphi)$ varies over the surface. By focusing on local quantities and differential equations we can make educated guesses for the general result. We will derive these results in the context of a general mathematical discussion later in this lecture.

Now let’s turn to differentiation. We saw in Eq. 7-9 that the derivatives of vectors tangent to the sphere are typically not tangent to the sphere. In fact, the definition of a geodesic reads that the derivative of the tangent to the geodesic in the direction of the geodesic should be normal to the sphere – it should have no components in the local tangent plane. We will motivate and
discuss this fact later in this lecture. It is also discussed in the textbook and in other supplementary lectures, especially #8. This fact leads to the definition of the “covariant” derivative of a tangent vector field on the sphere: the covariant derivative is the projection onto the tangent plane of the derivative itself. With more precision, we consider a curve on the surface \( \mathbf{r}(t) \). Then the covariant derivative in the direction of increasing \( t \) reads,

\[
\frac{d\mathbf{r}}{dt} = \mathbf{r}_t - (\mathbf{r}_t \cdot \mathbf{N})\mathbf{N}
\]

where \( \mathbf{N} \) is the local unit normal to the sphere and \( \mathbf{r}_t = \frac{\partial \mathbf{r}}{\partial t} \). The curve might be specified on \( S^2 \) by writing components in the \((\theta, \phi)\) basis, \( \mathbf{r}(t) = (\theta(t), \phi(t)) \), for example.

The covariant derivative is an intrinsic operation on the surface and can be defined without reference to \( \mathbb{R}^3 \), however Eq. 20 is appropriate for classical differential geometry and will be used exclusively here. In fact, the covariant derivative can be used as the source of other intrinsic properties of the geometry of the surface, such as the geodesic curvature \( \kappa_g \) of a path and the Gaussian curvature \( K \) of the surface. We have discussed \( \kappa_g \) in supplementary lecture #8 where we also discussed parallel transport. Recall that a vector field is said to be parallel transported along a curve if its covariant derivative vanishes everywhere along the curve. Let’s review the relationship between parallel transport and surface curvature in the case of the sphere. Consider the vector \( \mathbf{v}(t) \) parallel transported along the geodesic shown in Fig. 16,
Along the AB segment of the geodesic path, $\vec{v}(t)$ is the tangent to the path as shown. In segment BC, the equator, $\vec{v}(t)$ is perpendicular to the direction of the path and points directly south. Then along segment CA, back to the north pole, $\vec{v}(t)$ points along the tangent vector of the geodesic again. So, $D\vec{v}(t)/\partial t = 0$ everywhere along the path. It is parallel transported. From the perspective of the intrinsic geometry of the surface, $\vec{v}(t)$ maintains a constant orientation to the local tangent plane all along the path ABC. The surprise is now that $\vec{v}$ at the end of its journey is different from $\vec{v}$ at the beginning – its orientation has changed by the opening angle $\varphi$ shown in Fig. 17. But $\varphi$ is governed by the Gauss-Bonnet Theorem,

$$\varphi = \iint KdA = \frac{1}{\mathcal{R}^2} \Delta$$

where $\Delta$ is the area of the triangle ABC. So, $\varphi$ is a measure of the average curvature inside $\Delta$.

This result suggests that if the surface were curved, but otherwise arbitrary, and we considered an infinitesimal $\Delta$ so we isolate a region of fixed $K$, then the deficit angle $\varphi$ will be a measure of the local curvature $K$. 
We learn that parallel transportation is path dependent. So, if we parallel transport a vector from point A to point B on a surface through different paths, the results will be different. But this means that if we covariantly differentiate a vector in direction 1 and then in direction 2, we will get a different result than if we differentiate first in the direction 2 and then in the direction 1. On a flat surface where $K = 0$ the path dependence will disappear.

Let’s work through an example on the sphere to make this point in a familiar setting doing elementary and explicit differentiations before turning to general considerations later in this lecture. Consider a point $\vec{r}(\theta, \varphi)$ on a unit sphere. Later we will generalize to a sphere of radius $R$ by dimensional analysis. This will be essential to identifying the curvature $K$ in the final results. We begin with a unit sphere to reduce the writing temporarially. Recall,

$$\vec{r}(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$$

At each $(\theta, \varphi)$ there is a tangent plane spanned by $\vec{r}_\theta = \partial \vec{r} / \partial \theta$ and $\vec{r}_\varphi = \partial \vec{r} / \partial \varphi$,

$$\vec{r}_\theta = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta)$$

$$\vec{r}_\varphi = (-\sin \theta \sin \varphi, \sin \theta \cos \varphi, 0)$$

Note that $|\vec{r}_\theta|^2 = 1, |\vec{r}_\varphi|^2 = \sin^2 \theta, \vec{r}_\theta \cdot \vec{r}_\varphi = 0$.

We want to calculate the commutator,

$$\frac{D}{\partial \varphi} \frac{D}{\partial \theta} - \frac{D}{\partial \theta} \frac{D}{\partial \varphi} = \left[ \frac{D}{\partial \varphi}, \frac{D}{\partial \theta} \right]$$

and verify that it is non-zero and proportional to $K$. If we can do this for the basis $\vec{r}_\theta$ and $\vec{r}_\varphi$, we can get general results by linear superposition. In order to calculate $D \vec{r}_\theta / \partial \theta$ we first need,

$$\frac{\partial}{\partial \theta} \vec{r}_\theta = (-\sin \theta \cos \varphi, -\sin \theta \sin \varphi, -\cos \theta) = -\vec{N}$$

where we identified the unit normal. We learn that

$$\frac{D}{\partial \theta} \vec{r}_\theta = 0$$

which is just the statement that $\vec{r}(\theta, \varphi = \varphi_0)$ for variable $\theta$ and fixed $\varphi = \varphi_0$ is a geodesic, a great circle. We knew this already on general grounds and the algebra confirms it. Next we need,
\[
\frac{\partial}{\partial \varphi} \vec{r}_\theta = (-\cos \theta \sin \varphi, \cos \theta \cos \varphi, 0)
\]

We check that \( \vec{N} \cdot \vec{r}_{\theta \varphi} = 0 \), so \( D\vec{r}_\theta / d\varphi = \partial \vec{r}_\theta / \partial \varphi \). Finally, we calculate \( \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} \vec{r}_\theta = (\sin \theta \sin \varphi, -\sin \theta \cos \varphi, 0) \) and check that this vector is perpendicular to \( \vec{N} \) so, \( \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} \vec{r}_\theta = \frac{D}{\partial \theta} \frac{D}{\partial \varphi} \vec{r}_\theta \). Collecting these results,

\[
\left[ \frac{D}{\partial \varphi}, \frac{D}{\partial \theta} \right] \vec{r}_\theta = -(\sin \theta \sin \varphi, -\sin \theta \cos \varphi, 0) = \vec{r}_\varphi
\]

Clearly this case is “special” since \( \vec{r}(\theta, \varphi = \varphi_0) \) is a geodesic, so the right hand side of Eq. 22 is simpler than the general case. Our next calculation will expose the geometry better. In anticipation of that, note that \( \vec{r}_\varphi = (\vec{r}_\theta \times \vec{r}_\varphi) \times \vec{r}_\theta \) and write Eq. 22 as,

\[
\left[ \frac{D}{\partial \varphi}, \frac{D}{\partial \theta} \right] \vec{r}_\theta = K (\vec{r}_\theta \times \vec{r}_\varphi) \times \vec{r}_\theta
\]

where we have restored factors of \( R \) and identified \( K = 1/R^2 \). This form of the result will prove to be general as we shall show in a later section of this lecture.

Now we turn to \( \vec{r}_\varphi \). The calculations of the derivatives are similar to that above, but since \( \vec{r}_\varphi \) is not a tangent to a geodesic, the arithmetic is more elaborate and the second term in Eq. 20 must be evaluated in several steps. After doing the algebra explicitly we find,

\[
\left[ \frac{D}{\partial \varphi}, \frac{D}{\partial \theta} \right] \vec{r}_\varphi = \sin^2 \theta \vec{r}_\theta
\]

But we should be able to write the right hand side of this expression just in terms of the relevant vectors, \( \vec{r}_\theta \) and \( \vec{r}_\varphi \). We noticed earlier that \( |\vec{r}_\varphi|^2 = \sin^2 \theta \) and since \{\( \vec{r}_\theta, \vec{r}_\varphi \)\} is an orthogonal set, it is easy to verify \( (\vec{r}_\varphi \times \vec{r}_\theta) \times \vec{r}_\varphi = |\vec{r}_\varphi|^2 \vec{r}_\theta \). Lastly, generalizing to a sphere of radius \( R \) (curvature \( K = 1/R^2 \)), we have,

\[
\left[ \frac{D}{\partial \varphi}, \frac{D}{\partial \theta} \right] \vec{r}_\varphi = K (\vec{r}_\varphi \times \vec{r}_\theta) \times \vec{r}_\varphi
\]

which is the form of the result that will generalize to a general curved surface where \( K \) varies from point to point.
We have emphasized the importance of Eq. 23-24 because of their importance in Riemannian geometry. The reader should consult the textbook for discussions of parallel transport, covariant differentiation and their relation to the Riemann curvature tensor.

Geometry of Curved Surfaces
The Metric

The basis of the intrinsic geometry of a curved surface is the metric. Suppose we have a coordinate mesh \((u, v)\) on the surface so points are labeled \(\vec{r}(u, v)\). If two points P and Q are infinitesimally close, there is a vector distance \(d\vec{r}\) between them and in this coordinate system \(d\vec{r} = \vec{r}_u du + \vec{r}_v dv\) where \(\vec{r}_u = \partial \vec{r} / \partial u\) and \(\vec{r}_v = \partial \vec{r} / \partial v\). Then the distance squared between P and Q is

\[
d\vec{r}^2 = d\vec{r} \cdot d\vec{r} = (\vec{r}_u du + \vec{r}_v dv) \cdot (\vec{r}_u du + \vec{r}_v dv) = E du^2 + 2F dudv + G dv^2
\]

where,

\[
E = \vec{r}_u \cdot \vec{r}_u \\
F = \vec{r}_u \cdot \vec{r}_v \\
G = \vec{r}_v \cdot \vec{r}_v
\]

Using these expressions we can calculate distances and angles between vectors on the surface. For example, suppose we have a path \(\vec{r}(t) = (u(t), v(t))\) on the surface. Then the length of the path from a parameter value \(t = 0\) to \(t\) reads,

\[
s = \int |\vec{r}(t)| dt = \int \left| \frac{d\vec{r}}{dt} \right| dt = \int \sqrt{\frac{d\vec{r} \cdot d\vec{r}}{dt^2}} dt = \int_0^t (Eu'^2 + 2Fu'v' + Gv'^2)^{1/2} dt
\]

where “prime” means differentiation with respect to the parameter \(t\).

We can also calculate the surface area of regions on the surface. We consider a parallelepiped with edges \(\vec{r}_u du\) and \(\vec{r}_v dv\) and area \(dA = |\vec{r}_u du \times \vec{r}_v dv| = |\vec{r}_u \times \vec{r}_v| dudv\). Then the area of a finite surface region is,

\[
A = \iint |\vec{r}_u \times \vec{r}_v| dudv
\]

This can be written in terms of the metric by observing that,

\[
|\vec{r}_u \times \vec{r}_v|^2 + |\vec{r}_u \cdot \vec{r}_v|^2 = |\vec{r}_u|^2 |\vec{r}_v|^2 (\sin^2 \theta + \cos^2 \theta) = |\vec{r}_u|^2 |\vec{r}_v|^2
\]
where \( \theta \) is the angle between \( \mathbf{r}_u \) and \( \mathbf{r}_v \). So, 
\[
|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{|\mathbf{r}_u|^2 |\mathbf{r}_v|^2 - |\mathbf{r}_u \cdot \mathbf{r}_v|^2} = \sqrt{EG - F^2}.
\]
Therefore, the surface area can be written,
\[
A = \iiint \sqrt{EG - F^2} \, du \, dv
\]

Finally we will need to consider vectors in the vicinity of a point \( P \) on the surface. A natural 3 dimensional basis there consists of the two tangent vectors \( \mathbf{r}_u \) and \( \mathbf{r}_v \) and their normal,
\[
\mathbf{N} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \frac{\mathbf{r}_u \times \mathbf{r}_v}{\sqrt{EG - F^2}}
\]

We choose orientations so the members of the basis \( \{\mathbf{r}_u, \mathbf{r}_v, \mathbf{N}\} \) have the same relationship to one another as the Cartesian basis \( \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \) does for \( \mathbb{R}^3 \). Then the surface is oriented in the vicinity of the point \( P \).

**The Gauss Map and Its Derivative, the Weingarten Map**

In order to study the shapes and curvatures of surfaces, Gauss imagined covering them with pins, Fig. 17: at each point on the surface he constructed a unit normal from vectors in the tangent space, \( \mathbf{r}_u \) and \( \mathbf{r}_v \),
\[
\mathbf{N}(p) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}(p)
\]

The association of the point \( p \) with \( \mathbf{N}(p) \) is called the Gauss map.
Next let there be a curve on the surface, $\mathbf{r}(t)$. Then at every $t$ there is a tangent to the curve, $d\mathbf{r}/dt$. The surface is also covered with normal vectors $\mathbf{N}(u, v)$, so one can calculate $\mathbf{N}(t)$ along the curve, as well as its derivatives, the Weingarten map, $d\mathbf{N}/dt = \mathbf{N}'(t)$. Since $\mathbf{N} \cdot \mathbf{N} = 1$, by differentiation, $\mathbf{N} \cdot \mathbf{N}' = 0$, so $\mathbf{N}'$ lies in the tangent plane. Then applying the chain rule to the curve $\mathbf{r}(t) = (u(t), v(t))$,

$$\mathbf{N}'(t) = \mathbf{N}_u u'(t) + \mathbf{N}_v v'(t)$$

So, $\mathbf{N}'(t)$ is a linear operator on the vectors of the tangent plane at $t$, $\mathbf{r}'(t) = (u'(t), v'(t))$. To make this observation useful, we can write $\mathbf{N}'(t)$ in terms of the basis $(\mathbf{r}_u, \mathbf{r}_v)$. We will establish below that $\mathbf{N}'(t)$ is a self-adjoint operator which can be represented as a $2 \times 2$ matrix which can be diagonalized. Recall from linear algebra that the eigen-directions of such a matrix will be perpendicular. Call them $\{\mathbf{e}_1, \mathbf{e}_2\}$. Each has an eigenvalue $\{k_1, k_2\}$ associated with it. To show that $\mathbf{N}'(t)$ is self adjoint, we need to show that it satisfies,

$$\mathbf{N}_u \cdot \mathbf{r}_v = \mathbf{N}_v \cdot \mathbf{r}_u$$

Fig. 17 The Gauss map: the unit normals to the surface populate the unit sphere
This follows from \( \vec{N} \cdot \vec{r}_u = \vec{N} \cdot \vec{r}_v = 0 \) by differentiation,
\[
\vec{N}_v \cdot \vec{r}_u + \vec{N} \cdot \vec{r}_{uv} = 0, \quad \vec{N}_u \cdot \vec{r}_v + \vec{N} \cdot \vec{r}_{vu} = 0 \tag{33}
\]
Since \( \vec{r}_{uv} = \vec{r}_{vu} \), we have the result, Eq. 32. Now we can define a quadratic form, \( -d\vec{N}(t) \cdot d\vec{r}(t) \), which equals the projection of the curvature of \( \vec{r}(t) \) onto the normal \( \vec{N}(t) \) to the surface.

To verify this last point, choose the parameter \( t \) to be the arclength of \( \vec{r}(s) \). Then since \( \vec{N}(s) \cdot \vec{r}'(s) = 0 \), we learn, by differentiation, that \( \vec{N}(s) \cdot \vec{r}''(s) = -\vec{N}'(s) \cdot \vec{r}'(s) \). So,
\[
-d\vec{N}(s) \cdot d\vec{r}(s) = -\vec{N}'(s) \cdot \vec{r}'(s) \ ds^2 = \vec{N}(s) \cdot \vec{r}''(s) ds^2 = \vec{N}(s) \cdot \kappa(s) \vec{n}(s) ds^2 = \kappa_n(s) ds^2
\]
where we recalled the definition of the curvature of \( \vec{r}(s) \), \( \vec{r}''(s) = \kappa(s) \vec{n}(s) \), where \( \vec{n}(s) \) is the unit normal to \( \vec{r}(s) \), as discussed above, in the textbook and supplementary lecture #8. Here we will write \( \kappa(s) \vec{n}(s) = \kappa(s) \cos \theta = \kappa_n(s) \) and \( \theta(s) \) is the angle between \( \vec{N}(s) \) and \( \vec{n}(s) \). \( \kappa_n(s) \) is the component of the curvature of \( \vec{r}(s) \) that is normal to the surface there.

The quantity \( -d\vec{N} \cdot d\vec{r} \) is the “second fundamental form” of the surface. If we choose \( \vec{r}'(s) = \vec{e}_1 \), the eigen-direction associated with the eigenvalue \( k_1 \), then \( -\vec{N}'(s) = k_1 \vec{e}_1 \).

Similarly, choosing \( \vec{r}'(s) = \vec{e}_2 \) then, \( -\vec{N}'(s) = k_2 \vec{e}_2 \). Finally, recall from linear algebra that \( k_1 \) and \( k_2 \) (\( k_1 \geq k_2 \)) are the maximum and minimum of the second fundamental form restricted to the unit circle of vectors on the tangent plane. If we choose an arbitrary direction in the tangent plane, \( \vec{v} = \vec{e}_1 \cos \theta + \vec{e}_2 \sin \theta \), then
\[
\kappa_n(\vec{v}) = -\vec{N}'(\vec{v}) \cdot \vec{v} = -\vec{N}'(\vec{e}_1 \cos \theta + \vec{e}_2 \sin \theta) \cdot (\vec{e}_1 \cos \theta + \vec{e}_2 \sin \theta) \nonumber
\]
\[
\kappa_n(\vec{v}) = (k_1 \vec{e}_1 \cos \theta + k_2 \vec{e}_2 \sin \theta) \cdot (\vec{e}_1 \cos \theta + \vec{e}_2 \sin \theta) = k_1 \cos^2 \theta + k_2 \sin^2 \theta \quad \tag{34}
\]
The eigen-directions of \( -\vec{N}'(s) \) will prove to be very informative. These directions are sometimes call “principal directions” and curves which have them as their tangents are called “principal curves” or “lines of principal curvature”.

Now let’s turn to the curvature of the surface. To begin, recall that the determinant and the trace of a linear transformation such as \( -\vec{N}' \) are both independent of the basis \{\( \vec{e}_1, \vec{e}_2 \)\}. The determinant, which is just the product \( k_1 k_2 \) in the \{\( \vec{e}_1, \vec{e}_2 \)\} basis, is the famous Gaussian
curvature $K$, and the trace, or rather half the trace, $\frac{1}{2}(k_1 + k_2)$, in the $\{\vec{e}_1, \vec{e}_2\}$ basis, is the mean curvature $H$,

$$K = k_1 k_2, \quad H = \frac{1}{2}(k_1 + k_2)$$

In the case of the plane $k_1 = k_2 = 0$ and for the sphere of radius $\mathcal{R}$, $k_1 = k_2 = 1/\mathcal{R}$.

These basics give us the geometric idea of $\kappa_n$, $K$ and $H$, but to expedite explicit calculations in general and specific cases, we need to develop more formalism. To begin, write the linear transformation $\mathbf{N} \cdot \mathbf{r}'(t) = \mathbf{N} \cdot \mathbf{r}(t) + \mathbf{N} \cdot \mathbf{r}'(t)$ in the basis $\{\mathbf{r}_u, \mathbf{r}_v\}$,

$$\mathbf{N}_u = a_{11}\mathbf{r}_u + a_{21}\mathbf{r}_v, \quad \mathbf{N}_v = a_{12}\mathbf{r}_u + a_{22}\mathbf{r}_v$$

So,

$$\mathbf{N}' = (a_{11}\mathbf{r}_u + a_{21}\mathbf{r}_v)u' + (a_{12}\mathbf{r}_u + a_{22}\mathbf{r}_v)v' = (a_{11}u' + a_{12}v')\mathbf{r}_u + (a_{21}u' + a_{22}v')\mathbf{r}_v$$

Then $\mathbf{N}'(t)$ can be written in matrix notation,

$$[\mathbf{N}'(t)]^\top \mathbf{r}'(t) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

But we can write the second fundamental form in the basis $\{\mathbf{r}_u, \mathbf{r}_v\}$,

$$-(\mathbf{N}_u du + \mathbf{N}_v dv) \cdot (\mathbf{r}_u du + \mathbf{r}_v dv) = -\mathbf{N}_u \cdot \mathbf{r}_u du^2 - (\mathbf{N}_v \cdot \mathbf{r}_u + \mathbf{N}_u \cdot \mathbf{r}_v)du dv - \mathbf{N}_v \cdot \mathbf{r}_v dv^2$$

$$= edu^2 + 2f du dv + gdv^2$$

where we have introduced the coefficients $e, f$ and $g$ and have determined,

$$e = -\mathbf{N}_u \cdot \mathbf{r}_u = \mathbf{N} \cdot \mathbf{r}_{uu}$$

$$f = -\mathbf{N}_v \cdot \mathbf{r}_u = \mathbf{N} \cdot \mathbf{r}_{uv} = -\mathbf{N}_u \cdot \mathbf{r}_v$$

$$g = -\mathbf{N}_v \cdot \mathbf{r}_v = \mathbf{N} \cdot \mathbf{r}_{vv}$$

where we have observed that $-\mathbf{N}_u \cdot \mathbf{r}_u = \mathbf{N} \cdot \mathbf{r}_{uu}$, which follows upon differentiating $\mathbf{N} \cdot \mathbf{r}_u = 0$: $\frac{\partial}{\partial u}(\mathbf{N} \cdot \mathbf{r}_u) = 0 = \mathbf{N}_u \cdot \mathbf{r}_u + \mathbf{N} \cdot \mathbf{r}_{uu}$ and similarly for the other cases. But the coefficients of the
matrix representation of $\vec{\mathcal{N}}'$, $\{a_{ij}\}$, involved $\vec{\mathcal{N}}_u, \vec{\mathcal{N}}_v, \vec{r}_u$ and $\vec{r}_v$, so we can solve $\{a_{ij}\}$ in terms of the coefficients of the first and second fundamental forms. In particular, from Eq. 36,

\begin{align*}
-e &= \vec{N}_u \cdot \vec{r}_u = a_{11}E + a_{21}F \\
-g &= \vec{N}_v \cdot \vec{r}_u = a_{12}F + a_{22}G \\
-f &= \vec{N}_u \cdot \vec{r}_v = a_{11}F + a_{21}G \\
-f &= \vec{N}_v \cdot \vec{r}_u = a_{12}E + a_{22}F
\end{align*}

where we identified $E = \vec{r}_u \cdot \vec{r}_u$, $F = \vec{r}_u \cdot \vec{r}_v$ and $G = \vec{r}_v \cdot \vec{r}_v$. But Eq. 40 can be written in a more manageable matrix form,

\[
-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}
\]

So, the matrix representation of $\vec{\mathcal{N}}'$ operating on the matrix representation of the first fundamental form yields the second fundamental form. We can solve for $\{a_{ij}\}$,

\[
\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}
\]

The inverse of a symmetric $2 \times 2$ matrix is easy to compute,

\[
\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG-F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}
\]

Then carrying out the matrix multiplications in Eq. 42, we solve for $\{a_{ij}\}$,

\begin{align*}
a_{11} &= \frac{fF-eG}{EG-F^2} \\
a_{12} &= \frac{gF-fG}{EG-F^2} \\
a_{21} &= \frac{eF-fE}{EG-F^2} \\
a_{22} &= \frac{fF-gE}{EG-F^2}
\end{align*}

Now we have the geometric quantities,

\[
K = \det a_{ij} = \frac{eg-f^2}{EG-F^2}
\]

So the Gaussian curvature is the ratio of the determinants of the second and first fundamental forms. In addition,
\[ H = -\frac{1}{2} (a_{11} + a_{22}) = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2} \]

Illustrations. Parametrizing Surfaces and Computing their Properties

After all this algebra, let’s look at some examples to understand the results better. First consider a torus. It is a surface of revolution, so we begin with a circle of radius \( r \) translated by a distance \( a \) along the x-axis. This circle, which in Cartesian coordinates reads,

\[ (a + r \cos \theta, 0, r \sin \theta) \]

for \( 0 < \theta < 2\pi \), is shown in Fig 18,

Fig. 18 The construction of a torus as a surface of revolution

The torus is made by rotating the circle around the z-axis,

\[ \vec{r}(\theta, \varphi) = ((a + r \cos \theta) \cos \varphi, (a + r \cos \theta) \sin \varphi, r \sin \theta) \]
where $\varphi$ is the angle in the x-y plane and we must take $a > r$. Each point P sweeps out a circle of radius $(a + r \cos \theta)$ around the z-axis. It is clear that the coordinate lines, $\tilde{r}(\theta = \theta_0, \varphi)$ and $\tilde{r}(\theta, \varphi = \varphi_0)$ provide an orthogonal coordinate mesh on the torus.

Now it is straight-forward to calculate the quantities $\tilde{r}_\theta$, $\tilde{r}_\varphi$, $\tilde{r}_{\theta\theta}$, $\tilde{r}_{\theta\varphi}$ and $\tilde{r}_{\varphi\varphi}$ to compute the coefficients $e, f, g, E, F$ and $G$ of the first and second fundamental forms. We leave the details to the reader and quote the results,

$$E = r^2, F = 0, G = (a + r \cos \theta)^2, e = r, f = 0, g = \cos \theta (a + r \cos \theta)$$

The first fundamental form reads,

$$d\tilde{r}^2 = r^2 d\theta^2 + (a + r \cos \theta)^2 d\varphi^2$$

which we could have inferred without calculation: $rd\theta$ is the line element around the circle of radius $r$ and $(a + r \cos \theta) d\varphi$ is the line element along the circles that are swept out when the point P is rotated around the z axis. Finally we can calculate the Gaussian curvature $K = \frac{eg-f^2}{EG-F^2}$ finding,

$$K = \frac{\cos \theta}{r(a+r \cos \theta)}$$

We note that $K > 0$ for $-\pi/2 < \theta < \pi/2$ and $K < 0$ for $\pi/2 < \theta < 3\pi/2$. $K$ is the product of the principal curvatures, $k_1$ and $k_2$. One principal curve is the circle of radius $r$. It’s curvature is $1/r$ which points normal to the torus and so $k_1 = 1/r$. The other principal curve is the circle of radius $(a + r \cos \theta)$ and its curvature points perpendicular to the z axis as shown in Fig. 19,
The projection of $\kappa \vec{n} = (a + r \cos \theta)^{-1} \vec{n}$ onto $\vec{N}$ clearly brings in a factor of $\vec{n} \cdot \vec{N} \cos \theta$, so $k_2 = \cos \theta / (a + r \cos \theta)$. So, $K = k_1 k_2$ reproduces Eq. 48 with much less(!) work.

We also understand that $K = 0$ at the top and bottom of the torus where $\theta = \pi/2$ or $\theta = -\pi/2$: in those cases $\kappa \vec{n} = (a + r \cos \theta)^{-1} \vec{n}$ is perpendicular to $\vec{N}$ so $k_2 = 0$. Also, $K > 0$ for $-\pi/2 < \theta < \pi/2$ because then $k_1$ and $k_2$ are positive, however, $K < 0$ for $\pi/2 < \theta < 3\pi/2$ because then $k_1$ is positive but $k_2$ is negative: its $\vec{n}$ points radially out from the $z$ axis but $\vec{N}$ points toward the $z$ axis, $\vec{n} \cdot \vec{N} = \cos \theta$ which is negative for $\pi/2 < \theta < 3\pi/2$.

We also note that the tangent plane to the torus for $P$ at $-\pi/2 < \theta < \pi/2$ where $K > 0$ is on one side of the torus, but for $P$ at $\pi/2 < \theta < 3\pi/2$, where $K < 0$ there are points on the torus on both sides of the tangent plane. This behavior is more general than this example and will be discussed below when we discuss the geometric interpretation of the second fundamental form.
Now let’s consider general surfaces of revolution. We begin with a curve in the x-z plane parametrized by its arclength s,

\[(\rho(s), 0, h(s))\]

where \(\rho(s) > 0\) and \(\rho'(s)^2 + h'(s)^2 = (d\rho^2 + dh^2)/ds^2 = 1\) as shown in Fig. 20,

![Fig. 20 Rotating a curve in the x-z plane around the z axis](image)

When the curve is rotated around the z axis, a surface is generated,

\[\vec{r}(s, \varphi) = (\rho(s) \cos \varphi, \rho(s) \sin \varphi, h(s))\]

where \(0 < \varphi < 2\pi\). Now it is easy to calculate the various derivatives of \(\vec{r}(s, \varphi)\), \((\vec{r}_s, \vec{r}_\varphi, \vec{r}_{ss}, \vec{r}_{s\varphi}, \vec{r}_{\varphi\varphi})\), and calculate the coefficients of the first and second fundamental forms,

\[E = \rho^2, F = 0, G = \rho'^2 + h'^2 = 1, \quad e = -\rho h', f = 0, \quad g = \rho''h' - \rho'h''\]

So, the metric is \(d\vec{r}^2 = \rho^2 d\varphi^2 + ds^2\). Since the line element in the \(\varphi\) direction is \(\rho d\varphi\), this metric Eq. 50 is as expected. The Gaussian curvature is,
\[ K = \frac{eg-f^2}{EG-F^2} = \frac{-h'(h''-\rho''\rho')}{\rho} \quad \text{(51.a)} \]

However, since \( \rho^2 + h^2 = 1 \) we can write \( K \) in terms of \( \rho \) alone. Differentiating \( \rho^2 + h^2 = 1 \) we learn \( \rho' \rho'' = -h'h'' \). Substituting into Eq. 51 we find,

\[ K = \frac{h''\rho' - h'h''}{\rho} = -\frac{h''\rho' + \rho''\rho'}{\rho} = -\frac{\rho''}{\rho} \quad \text{(51.b)} \]

Eq. 51.b allows us to calculate \( K \) easily.

We can look at this equation from another perspective: \( K \) satisfies the differential equation,

\[ \frac{d^2\rho}{ds^2} + K(s, \varphi)\rho = 0 \quad \text{(51.c)} \]

So, suppose we are given \( K(s, \varphi) \). Then we can predict the shape of the surface!

Next, since \( f = F = 0 \), the formulas for \( K \) and \( H \), the mean curvature, simplify to,

\[ K = \frac{eg}{EG} \quad H = \frac{1}{2} \left( \frac{e}{E} + \frac{g}{G} \right) \quad \text{(52.a)} \]

But \( K = k_1k_2 \) and \( H = \frac{1}{2}(k_1 + k_2) \), so we read off the principal curvatures of the surface,

\[ k_1 = \frac{e}{E} = -\frac{h'}{\rho} \quad k_2 = \frac{g}{G} = h'\rho'' - h''\rho' \quad \text{(52.b)} \]

Finally, let’s consider a surface written as a graph,

\[ \vec{r}(x, y) = (x, y, h(x, y)) \quad \text{(53)} \]

Then it is easy to calculate the derivatives,

\[ \vec{r}_x = (1, 0, h_x), \quad \vec{r}_y = (0, 1, h_y), \quad \vec{r}_{xx} = (0, 0, h_{xx}), \quad \vec{r}_{xy} = (0, 0, h_{xy}), \quad \vec{r}_{yy} = (0, 0, h_{yy}) \]

We read off the coefficients of the first fundamental form,

\[ E = \vec{r}_x \cdot \vec{r}_x = 1 + h_x^2 \quad F = \vec{r}_x \cdot \vec{r}_y = h_xh_y \quad G = \vec{r}_y \cdot \vec{r}_y = 1 + h_y^2 \]

Next, we calculate the normal and the coefficients of the second fundamental form,

\[ \vec{N}(x, y) = \begin{pmatrix} -h_x, -h_y, 1 \\ 1 + h_x^2 + h_y^2 \end{pmatrix}^{1/2} \]
Let’s orient the x-y plane to expose the geometry of the surface and find a simple geometric interpretation of the second fundamental form: Place the origin (0,0) at the point P of the tangent plane to the surface and take the tangent plane to be the x-y plane as shown in Fig. 21.

With these choices $h(0,0) = 0$, $h_x(0,0) = h_y(0,0) = 0$. If we Taylor expand $h(x, y)$ in the vicinity of point P,

\[
h(x, y) = \frac{1}{2} \left( h_{xx}(0,0)x^2 + 2h_{xy}(0,0)xy + h_{yy}(0,0)y^2 \right) + \ldots
\]
Let’s choose x along the direction of principal curvature \( k_1 \) so the y axis is along the second curvature direction. In this basis \( \{a_{ij}\} \) is diagonalized, so \( f = 0 \), (see Eq. 44, for example) which implies in turn that \( h_{xy}(0,0) = 0 \). Now,

\[
h(x, y) = \frac{1}{2} \left( h_{xx}(0,0)x^2 + h_{yy}(0,0)y^2 \right) + \cdots
\]

But we also observed in Eq. 52b that if \( f = F = 0 \), then in general,

\[
k_1(0,0) = \frac{e}{E} \quad k_2 = \frac{g}{G}
\]

But in this application \( e/E = e = h_{xx}(0,0) \) and \( g/G = g = h_{yy}(0,0) \). Therefore,

\[
h(x, y) = \frac{1}{2} \left( k_1 x^2 + k_2 y^2 \right) + \cdots
\]

We learn from this exercise that the second fundamental form tells us how quickly a surface deviates from its tangent plane!

In addition, Eq. 54 implies that if the surface has \( K > 0 \), so \( k_1 \) and \( k_2 \) have the same signs, then for \( x \) and \( y \) sufficiently small, the surface lies on one side of its tangent plane, but if \( K < 0 \), so \( k_1 \) and \( k_2 \) have different signs, some of the points on the surface lie on one side of the tangent plane and others lie on the other side. The surface near P therefore resembles a saddle for \( K < 0 \)!

This discussion generalizes observations we made about the torus.

**Geometric Interpretation of the Gauss Map**

Next we can review Gauss’ original geometric interpretation of \( K \). The Gauss map assigns a normal vector \( \vec{N}(p) \) to any point \( p \) on the surface. The area element on the surface of the unit sphere swept out by \( \vec{N}(p) \) reads,

\[
d\vec{N}_u \times d\vec{N}_v = \vec{N}_u du \times \vec{N}_v dv = \vec{N}_u \times \vec{N}_v \ dudv
\]

where \((u, v)\) parametrizes \( S^2 \) as well as the surface itself. To relate Eq. 55a to the area element on the original surface, we recall that \( \vec{N}_u = a_{11}\vec{r}_u + a_{12}\vec{r}_v \) and \( \vec{N}_v = a_{21}\vec{r}_u + a_{22}\vec{r}_v \), so Eq. 55a reads,

\[
(a_{11}\vec{r}_u + a_{12}\vec{r}_v) \times (a_{21}\vec{r}_u + a_{22}\vec{r}_v) \ dudv = (a_{11}a_{22} - a_{12}a_{21})\vec{r}_u \times \vec{r}_v \ dudv
\]
But $K = a_{11}a_{22} - a_{12}a_{21}$ and $dA = \vec{r}_u \times \vec{r}_v \; dudv$, the area element on the surface, so Eq. 55.a-b imply,

$$dA' = K \; dA$$

where $dA'$ is the area element on the unit sphere $S^2$. Therefore, the curvature is the ratio of these areas,

$$K = \lim_{\delta A \to 0} \frac{\delta A'}{\delta A}$$

Furthermore, $K$ carries the sign of $\vec{N}_u \times \vec{N}_v$ relative to $\vec{r}_u \times \vec{r}_v$. We can illustrate this in Fig. 22.a and 22.b,

![Diagram](image.png)

**Fig. 22a** Mapping the normals bounding an oriented region to the unit sphere, positive Gaussian curvature
In the case Fig. 22.a where $K > 0$ the curves are traversed in the same direction and the surfaces have the same orientation but for Fig. 22.b where $K < 0$ the curves are traversed in opposite directions and the surfaces have opposing orientations. These remarks are free of coordinate meshes. They are purely geometric in character.

Gauss’ Theorem Egregium. Mainardi-Codazzi Consistency Equations

The aim of this discussion is a demonstration that the Gaussian curvature is an intrinsic property of the surface: it can be calculated solely in terms of the metric and its derivatives. This means that $K$ is a “bending invariant”: surfaces which are related by bending without stretching, operations which preserve the metric locally, have identical Gaussian curvatures locally. For example, pieces of flat paper can be folded into right cylinders or cones. So, locally and away from any singularities (the tip of the cone, for example), these surfaces have identical metrics and $K$. 

Fig. 22.b Similar to Fig. 22.a, but for a surface with negative Gaussian curvature
We have derived this result, called Gauss’ Theorem Egregium (Excellent!) in the textbook by several methods. The discussion here supplements those discussions. The conceptual basis is to think of the triad \( \{ \vec{r}_u, \vec{r}_v, \vec{N} \} \) at each point \( p \) on the surface. This triad tells us how the tangent plane is oriented in \( R^3 \) and how the shape of the surface changes as \( p \) moves. We can express the derivatives of \( \{ \vec{r}_u, \vec{r}_v, \vec{N} \} \) in this basis

\[
\vec{r}_{uu} = \Gamma_{11}^1 \vec{r}_u + \Gamma_{11}^2 \vec{r}_v + e \vec{N}
\]

\[
\vec{r}_{uv} = \Gamma_{12}^1 \vec{r}_u + \Gamma_{12}^2 \vec{r}_v + f \vec{N}
\]

\[
\vec{r}_{vv} = \Gamma_{22}^1 \vec{r}_u + \Gamma_{22}^2 \vec{r}_v + g \vec{N}
\]

\[
\vec{N}_u = a_{11} \vec{r}_u + a_{21} \vec{r}_v
\]

\[
\vec{N}_v = a_{12} \vec{r}_u + a_{22} \vec{r}_v
\]

We have introduced the Christoffel symbols \( \Gamma_{ij}^k \) here. They are not(!) new parameters. They will be expressed in terms of the coefficients of the first fundamental form and their derivatives. We have also put the coefficients of the second fundamental form, \( e, f, \) and \( g, \) into Eq. 58. For example, the first equation in Eq. 58 can be projected onto the normal and then it reads \( \vec{r}_{uu} \cdot \vec{N} = e \) which we recognize as the definition of \( e. \) Taking the inner products of the first four equations with \( \vec{r}_u \) and \( \vec{r}_v, \) we find,

\[
\vec{r}_{uu} \cdot \vec{r}_u = \frac{1}{2} E_u = \Gamma_{11}^1 E + \Gamma_{11}^2 F
\]

\[
\vec{r}_{uu} \cdot \vec{r}_v = F_u - \frac{1}{2} E_v = \Gamma_{11}^1 F + \Gamma_{11}^2 G
\]

\[
\vec{r}_{uv} \cdot \vec{r}_u = \frac{1}{2} E_v = \Gamma_{12}^1 E + \Gamma_{12}^2 F
\]

\[
\vec{r}_{uv} \cdot \vec{r}_v = \frac{1}{2} G_u = \Gamma_{12}^1 F + \Gamma_{12}^2 G
\]

\[
\vec{r}_{vv} \cdot \vec{r}_u = F_v - \frac{1}{2} G_u = \Gamma_{22}^1 E + \Gamma_{22}^2 F
\]

\[
\vec{r}_{vv} \cdot \vec{r}_v = \frac{1}{2} G_v = \Gamma_{22}^1 F + \Gamma_{22}^2 G
\]
It is clear that we can solve the doublets of equations, Eq. 59.a, 59.b and 59.c for the Christoffel symbols $\Gamma^k_{ij}$ in terms of the first fundamental form and its derivatives. So, we learn that all geometric concepts that can be expressed in terms of Christoffel symbols are properties of the intrinsic geometry of the surface. The Christoffel symbols themselves depend on the coordinate mesh $(u, v)$ chosen. However, using the Christoffel symbols we will construct quantities that have geometric significance that are, in fact, independent of the coordinate mesh, such as $K$ and $\kappa_g$.

Our next exercise is to obtain relations between these quantities using the rather trivial looking “compatibility” expressions,

$$
\begin{align*}
(r_{uu})_v &= (r_{uv})_u , \quad (r_{vv})_u = (r_{uv})_v , \quad \vec{N}_{uv} = \vec{N}_{vu} \\
\end{align*}
$$

For example, if we write the first of these identities out using Eq. 58 and 59, we find,

$$
\begin{align*}
\Gamma^1_{11} r_{uv} + \Gamma^1_{12} r_{vu} + e\vec{N}_v + (\Gamma^1_{11})_v r_u + (\Gamma^2_{11})_v r_v + e_v \vec{N} &= \\
\Gamma^1_{12} r_{uu} + \Gamma^2_{12} r_{vu} + f\vec{N}_u + (\Gamma^1_{12})_u r_u + (\Gamma^2_{12})_u r_v + f_u \vec{N} \\
\end{align*}
$$

Then using Eq. 58 we can obtain an expression involving only the vectors $\vec{r}_u$, $\vec{r}_v$ and $\vec{N}$. The coefficients of each of these vectors must vanish when Eq. 61 is written in the form,

$$
A\vec{r}_u + B\vec{r}_v + C\vec{N} = 0
$$

Writing out the coefficient of $\vec{r}_v$ we have,

$$
\Gamma^1_{11}\Gamma^1_{12} + \Gamma^1_{11}\Gamma^2_{12} + e\alpha_{22} + (\Gamma^1_{11})_v = \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} + f\alpha_{21} + (\Gamma^2_{12})_u
$$

Substituting in the expressions for $\alpha_{ij}$ from Eq. 44 we find,

$$
(\Gamma^2_{12})_u - (\Gamma^1_{11})_v + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^1_{11}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^2_{22} = -E\frac{eg-f^2}{EG-F^2} = -KE
$$

Similarly, if we work on the $\vec{r}_u$ term in Eq.62 we learn,

$$
(\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} = KF
$$

We learn from these results that the Gaussian curvature $K$ depends only on the first fundamental form and its derivatives. It does not(!) depend on how the surface is embedded in
\( R^3 \). It only depends on the intrinsic properties of the surface itself. This is the content of Gauss’ Theorem Egregium which was derived in two other ways in the textbook. If we take Eq. 63 and write it out for orthogonal coordinates, \( F = 0 \), then we find, after using Eq. 59 to express the Christoffel symbols in terms of the first fundamental form and its derivatives,

\[
K = -\frac{1}{2\sqrt{EG}}\left\{ \left( \frac{E_u}{\sqrt{EG}} \right)_v + \left( \frac{g_u}{\sqrt{EG}} \right)_u \right\}
\]

which was also obtained in the textbook and is a central result of differential geometry.

To make contact with the terminology of classical differential geometry, the mapping \( \vec{N}(t) \to \vec{N}'(t) \), Eq. 44, is called the Weingarten map and the other consequences of the compatibility relations Eq. 60 are the Mainardi-Codazzi equations. Further explorations of these ideas reveals that this system of equations is complete. We are done. Further differentiations of Eq. 60 will not yield anything new [4].

Parallel Transport, Covariant Differentiation and the Gauss-Bonnet Theorem

Parallel Transport and Covariant Differentiation. Illustrations

We have already seen through illustrations and special cases that parallel transport of a vector is closely related to the curvature of the surface it is embedded in. In other supplementary lectures we defined the covariant derivative of a vector field \( \vec{v}(t) \) which lies in the tangent plane of a surface,

\[
\frac{D}{dt} \vec{v}(t) = \frac{\partial}{\partial t} \vec{v}(t) - \left( \frac{\partial}{\partial t} \vec{v}(t) \right) \cdot \vec{N}(t)
\]

where differentiation is done along a curve \( \vec{r}(t) \) which lies on the surface. Eq. 66 defines the covariant derivative as the components of \( \frac{\partial}{\partial t} \vec{v}(t) \) that lie in the tangent plane. These are the pieces of \( \frac{\partial}{\partial t} \vec{v}(t) \) that are intrinsic to the surface: an observer living in the surface is sensitive to the tangent plane \( \{\vec{r}_u, \vec{r}_v\} \) but not the normal \( \vec{N} \) to the surface.

This observation leads to the concept of “parallel” from the perspective of the intrinsic geometry of the surface. We say that if \( \frac{D}{dt} \vec{v}(t) = 0 \) along a curve \( \vec{r}(t) \), then \( \vec{v}(t) \) is “parallel
transported” along the curve. It should be clear that if the surface is the 2 dimensional plane, then covariant differentiation coincides with ordinary differentiation and parallel transport coincides with the usual notion of parallel transport as illustrated in Fig. 23,

![Fig. 23 Parallel transport along a planar curve](image)

On the plane when two vectors are parallel transported, their magnitudes and the angles between them are preserved. This is also the case for parallel transport of vectors along a curve embedded on a curved surface. To see this consider $\mathbf{v}(t) \cdot \mathbf{w}(t)$. The rate of change of this inner product along a curve $\mathbf{r}(t)$ is,

$$\frac{d}{dt} (\mathbf{v}(t) \cdot \mathbf{w}(t)) = \mathbf{v}'(t) \cdot \mathbf{w}(t) + \mathbf{v}(t) \cdot \mathbf{w}'(t) = \frac{D}{dt} \mathbf{v} \cdot \mathbf{w}(t) + \mathbf{v}(t) \cdot \frac{D}{dt} \mathbf{w} = 0$$

where we observed that $\frac{\partial}{\partial t} \mathbf{v}$ and $\frac{D}{dt} \mathbf{v}$ differ only by a component in the direction of $\mathbf{N}(t)$ and this difference doesn’t contribute to an inner product within the tangent plane.

Let’s illustrate parallel transport for a vector $\mathbf{w}(t)$ along a curve on a sphere which is a “cap” which is not a geodesic, a great circle. A cap is shown in Fig. 24,
How does $\vec{w}_0$ evolve by parallel transport along the cap C on the sphere? A clever way [1] to answer this question is to place a cone over the sphere so that the cone is tangent to the curve C as shown in Fig. 25a,
We can cut the cone along a meridian and unfold it to produce Fig. 25b,
The point is that once we unfold the cone, it is flat and parallel transport coincides with our familiar planar notion. This is shown in Fig. 25b where $\vec{w}_0$ has the same magnitude and direction as $\vec{w}(s)$. But we chose $\vec{w}_0$ to be parallel to the tangent to C at $s = 0$. However, after parallel transport through the arclength $s$, the tangent to C, $\vec{t}(s)$, and $\vec{w}(s)$ now have an angle between them of $\varphi$. We see that $\vec{t}(s)$ is not parallel transported along C. If we take the special case of $\theta = \pi/2$, so that C is the equator of the sphere, then $\vec{t}(s)$ is parallel transported around C and it remains parallel to $\vec{w}(s)$ for all $s$, $0 < s < 2\pi$. Recall from supplementary lecture #8 that $D\vec{t}(s)/ds = 0$ for a geodesic. This means that $\partial \vec{t}(s)/\partial s$ lies in the normal direction to the sphere and there is no variation of the vector in the tangent plane. In fact, we also discussed that the algebraic value of $D\vec{t}(s)/ds$ is the geodesic curvature of the curve C and the geodesic curvature is the projection onto the tangent place of the curvature $\kappa \vec{n}(s)$ of the curve $\vec{r}(s)$, so $\kappa \sin \theta(s) = \kappa_g$ where $\theta(s)$ is the angle between $\vec{n}$ and $\vec{N}$ at that point. We note that if $\kappa_g = 0$, then the curve C is “as straight as possible” on the surface S and so C is called a geodesic. These ideas were developed at length in supplementary lecture #8. They led to the relation,

$$\kappa^2 = \kappa_g^2 + \kappa_n^2$$

for the curvature of C and its projection in the tangent plane $\kappa_g$ and in the normal direction to the surface $\kappa_n$, as illustrated in Fig. 26.
The Turning angle for curves on Curved Surfaces and the Gauss-Bonnet Theorem

Now we need to generalize the idea of a “turning angle” from planar curves to curves embedded on surfaces. In a plane we can always label the orientation of a vector by taking its inner product with the unit vector \( \hat{i} \) in the x direction. The generalization of the “turning angle” from planar curves to curves embedded on surfaces will also give us a practical measure of the covariant derivative. In particular, consider two vector fields \( \vec{v} \) and \( \vec{w} \) tangent to a surface \( S \). Let \( \phi(p) \) be the angle between them at the point \( p \) on the surface, \( \vec{v}(p) \cdot \vec{w}(p) = \cos \phi(p) \) where we normalize both \( \vec{v} \) and \( \vec{w} \). We claim that \( d\phi/dt \) is the difference of the algebraic values of the covariant derivatives of \( \vec{v} \) and \( \vec{w} \) calculated along a curve \( \vec{r}(t) \),

\[
\left[ \frac{D\vec{w}}{dt} \right] - \left[ \frac{D\vec{v}}{dt} \right] = \frac{d\phi}{dt} \tag{69}
\]

This result follows from the inner product \( \vec{v}(t) \cdot \vec{w}(t) = \cos \phi(t) \) which we differentiate,
\[
\frac{d\bar{v}}{dt} \cdot \bar{w} + \bar{v} \cdot \frac{d\bar{w}}{dt} = -\sin \varphi \frac{d\varphi}{dt} \tag{70.a}
\]

Both \(\frac{d\bar{v}}{dt}\) and \(\frac{d\bar{w}}{dt}\) can be replaced by \(\frac{D\bar{v}}{\partial t}\) and \(\frac{D\bar{w}}{\partial t}\) in Eq. 70a because they are identical within the tangent plane where \(\bar{v}\) and \(\bar{w}\) live,

\[
\frac{D\bar{v}}{\partial t} \cdot \bar{w} + \bar{v} \cdot \frac{D\bar{w}}{\partial t} = -\sin \varphi \frac{d\varphi}{dt} \tag{70.b}
\]

We can make the directions of the covariant derivatives explicit,

\[
\frac{D\bar{v}}{\partial t} = \left[ \frac{D\bar{v}}{\partial t} \right] \bar{N} \times \bar{v} \quad , \quad \frac{D\bar{w}}{\partial t} = \left[ \frac{D\bar{w}}{\partial t} \right] \bar{N} \times \bar{w} \quad \tag{71.}
\]

The factor \(\bar{N} \times \bar{v}\) enforces the fact that the covariant derivative lies in the tangent plane and is perpendicular to \(\bar{v}\). Then,

\[
\frac{D\bar{v}}{\partial t} \cdot \bar{w} = \left[ \frac{D\bar{v}}{\partial t} \right] (\bar{N} \times \bar{v}) \cdot \bar{w} \quad , \quad \frac{D\bar{w}}{\partial t} \cdot \bar{v} = \left[ \frac{D\bar{w}}{\partial t} \right] (\bar{N} \times \bar{v}) \cdot \bar{w} = -\left[ \frac{D\bar{w}}{\partial t} \right] (\bar{N} \times \bar{v}) \cdot \bar{w}
\]

So,

\[
\frac{D\bar{v}}{\partial t} \cdot \bar{w} + \bar{v} \cdot \frac{D\bar{w}}{\partial t} = \left( \left[ \frac{D\bar{v}}{\partial t} \right] - \left[ \frac{D\bar{w}}{\partial t} \right] \right) (\bar{N} \times \bar{v}) \cdot \bar{w} = \left( \left[ \frac{D\bar{v}}{\partial t} \right] - \left[ \frac{D\bar{w}}{\partial t} \right] \right) (-\sin \varphi) \tag{72.}
\]

Substituting into Eq. 70.b, we derive Eq. 69.

An important special case of Eq. 69 follows from taking \(\bar{w}(s) = \bar{r}'(s)\), the unit tangent to the curve \(\bar{r}(s)\). Choose \(\bar{v}(s)\) to be a parallel transported field along \(C\) so \(D\bar{v}(s) / \partial s = 0\). Then Eq. 72 becomes,

\[
\kappa_g(s) = \left[ \frac{D\bar{v}'(s)}{\partial s} \right] = \frac{d\varphi}{dt} \tag{73.}
\]

This is a useful intuitive result: the geodesic curvature is the rate of change of the angle that the tangent to the curve makes with a parallel direction along the curve. Comparing this result to our earlier discussion of curves on a flat plane, we see that the geodesic curvature reduces to the usual curvature there.
Eq. 73 shows us that the geodesic curvature is intrinsic to the geometry of the surface $S$. We can use Eq. 73 to derive the Gauss-Bonnet theorem. The Gauss-Bonnet theorem was the subject of supplementary lecture #4, so we will not repeat a full discussion here, but we will derive the Gauss-Bonnet theorem for simple, smooth, closed curves. To do this we need the generalization of Eq. 69 when $\mathbf{w}(t) \rightarrow r'(t)$ and $\mathbf{v}(t) \rightarrow \tilde{n}_u(t)$. To begin consider Eq. 69 and use an orthogonal parametrization $\mathbf{r}(u, v)$. Then $\mathbf{e}_1 = \tilde{n}_u / \sqrt{E}$ and $\mathbf{e}_2 = \tilde{n}_v / \sqrt{G}$ form an orthonormal basis and $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{N}$. Then Eq. 69 becomes,

$$\left[ \frac{D\mathbf{r}'}{dt} \right] = \left[ \frac{D\mathbf{e}_1}{dt} \right] + \frac{d\phi}{dt}$$  

Let’s write the second term in terms of the coordinates $u(t)$ and $v(t)$. First,

$$\left[ \frac{D\mathbf{e}_1}{dt} \right] = \frac{d\mathbf{e}_1}{dt} \cdot (\mathbf{N} \times \mathbf{e}_1) = \frac{d\mathbf{e}_1}{dt} \cdot \mathbf{e}_2$$  

But using the chain rule we can write $d\mathbf{e}_1 / dt$ in terms of $du / dt$ and $dv / dt$, $d\mathbf{e}_1 / dt = \frac{\partial \mathbf{e}_1}{\partial u} \frac{du}{dt} + \frac{\partial \mathbf{e}_1}{\partial v} \frac{dv}{dt}$, so Eq. 75a becomes,

$$\frac{D\mathbf{e}_1}{dt} = \frac{\partial \mathbf{e}_1}{\partial u} \cdot \mathbf{e}_2 \frac{du}{dt} + \frac{\partial \mathbf{e}_1}{\partial v} \cdot \mathbf{e}_2 \frac{dv}{dt}$$  

These inner products can be written in terms of the first fundamental form,

$$\frac{\partial \mathbf{e}_1}{\partial u} \cdot \mathbf{e}_2 = \left( \frac{\mathbf{r}_u}{\sqrt{E}} \right)_u \cdot \left( \frac{\mathbf{r}_v}{\sqrt{G}} \right)_u = -\frac{1}{2} \frac{E_v}{\sqrt{EG}}$$  

Similarly,

$$\frac{\partial \mathbf{e}_1}{\partial v} \cdot \mathbf{e}_2 = \frac{G_u}{2 \sqrt{EG}}$$

Substituting Eq. 75-76 into Eq. 74, we find,

$$\left[ \frac{D\mathbf{r}'}{dt} \right] = \frac{1}{2 \sqrt{EG}} \left( G_u \frac{dv}{dt} - E_v \frac{du}{dt} \right) + \frac{d\phi}{dt}$$

which is the desired result. Taking the case $t \rightarrow s$ and identifying the geodesic curvature $\kappa_g(s) = \left[ \frac{D\mathbf{r}'}{dt} \right]$,

$$\kappa_g(s) = \frac{1}{2 \sqrt{EG}} \left( G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right) + \frac{d\phi}{ds}$$
This equation shows us that the curved surface \( S \) introduces another term to the planar relation between the turning angle and the geodesic curvature.

Eq. 78 leads to the Gauss-Bonnet Theorem. Integrate Eq. 78 along a closed, simple, regular curve \( C \) on the surface \( S \) as shown in Fig. 27,

\[
\int k_g(s) \, ds = \int \left( \frac{1}{2\sqrt{EG}} \left( G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right) \right) ds + \oint \frac{d\varphi}{ds} \, ds
\]

The first term on the right hand side of this equation is hiding the Gaussian curvature of the surface. To see this marvelous fact, use Green’s Theorem to write the line integral as a two dimensional integral over the inside region \( R \) enclosed by the curve \( C \) (Fig. 27),

\[
\oint (Adu + Bdv) = \iint \left( \frac{\partial B}{\partial u} - \frac{\partial A}{\partial v} \right) \, dudv
\]

which holds for smooth functions \( A(u, v) \) and \( B(u, v) \). Then,
\[ \oint (\frac{G_u}{2 \sqrt{EG}} \, dv - \frac{E_v}{2 \sqrt{EG}} \, du) = \oint \left[ \left( \frac{E_v}{2 \sqrt{EG}} \right)_v + \left( \frac{G_u}{2 \sqrt{EG}} \right)_u \right] \, dudv \]

\[ = \oint K \sqrt{EG} \, dudv = \oint K \, dA \]

where we identified the Gaussian curvature \( K \) from Eq. 65 and also identified the (surface) area element, \( \sqrt{EG} \, dudv = dA \) for \( F = 0 \). Finally, \( \varphi(s) \) turns through a total angle \( 2\pi \) as the curve \( C \) is traversed, so \( \oint \frac{d\varphi}{ds} \, ds = 2\pi \). Collecting everything,

\[ \oint \kappa_g(s) \, ds + \oint K \, dA = 2\pi \]

The implications of this result, one form of the Gauss-Bonnet Theorem, are enormous, as discussed and illustrated in supplementary lecture #4 where the theorem is also written down for piecewise smooth curves, surfaces with holes, etc. Applications of Eq. 79 to the global topology of surfaces and geodesics were considered there.

Geodesic Polar Coordinates and the Geodesic Deviation
For many applications in planar geometry and physics, polar coordinates are particularly effective. This coordinate system is shown in Fig. 28,

\[
x = \rho \cos \varphi \quad \quad \quad y = \rho \sin \varphi
\]

80.

The rays of constant \( \varphi = \varphi_0 \) are geodesics emanating from the origin \((x, y) = (0, 0)\). And circles of variable \( \varphi \), \(0 < \varphi < 2\pi\) and fixed \( \rho \) are called “geodesic circles”. The coordinate curves are orthogonal.

Now we are interested in the analog of this construction on a curved surface. We have a polar coordinate system on the tangent plane and map it to the surface in Fig. 29.
The geodesic polar coordinates on the surface $S$ are generated from the planar polar coordinates on the tangent plane, by thinking of the radial geodesics on the tangent plane as ropes which can be placed on the surface below (see Fig. 29). We have already seen an example of this.
construction, although it was described differently: this is polar coordinates on a sphere, Fig. 30,

Recall that on the unit sphere $\theta$ is the arclength of the radial geodesics emanating form the north pole and the “parallels”, the circles of constant $\theta$ and variable $\varphi$, $0 < \varphi < 2\pi$, have circumference $2\pi \sin \theta$ and coincide with a geodesic only for $\theta = \pi/2$. We have already seen in our discussions of spherical geometry that the discrepancy $2\pi \theta - 2\pi \sin \theta$ for small $\theta$ indicates that the Gaussian curvature of the sphere is $K = \mathcal{R}^{-2}$ where $\mathcal{R}$ is the radius on the sphere.

In the case of a general surface we define the Gaussian polar coordinates on the surface by mapping the radial lines of the polar coordinate system to the surface by preserving their arclength. Again, we think of the radial lines on the tangent plane as flexible ropes. In terms of the first fundamental form of the surface this means that $E = 1$ exactly. Next, the geodesic radial lines and the geodesic circles in Fig. 31 remain perpendicular to one another so $F = 0$. To motivate this result, consider two radial geodesics making up the coordinate mesh as shown in Fig. 31,
The two radial coordinate lines which are separated by an infinitesimal angle $\delta \varphi$ must have the same lengths at points 1 and 2. To guarantee this, the curve $C$, the geodesic circle, must be perpendicular to the radial geodesics. Finally, the coefficient $G$ of the first fundamental form must approach the corresponding coefficient of the planar polar coordinate system when $\rho$, the length of the geodesics, becomes small. So, $\sqrt{G} \to \rho$ as $\rho \to 0$, implying that $\lim_{\rho \to 0} \sqrt{G} = 0$ and $\lim_{\rho \to 0} \left( \sqrt{G} \right) \rho = 1$.

Next, $G$ can be related to the Gaussian curvature at point $p$ because when $E = 1$ and $F = 0$, the formula for $K$, Eq. 65, reduces to, after some algebra,

$$K = -\frac{(\sqrt{\tau})_{\rho\rho}}{\sqrt{\sigma}}$$

81.

We have seen this differential equation before when we discussed spherical geometry and we speculated that Eq. 81 would be more general.
If we differentiate Eq. 81 we learn more,
\[ \frac{\partial^3 \sqrt{G}}{\partial \rho^3} = -K(\sqrt{G})_{\rho} - K_{\rho} \sqrt{G} \]  
82.a

The value of \( K \) at \( p \) follows, using the fact that \( \lim_{\rho \to 0} \sqrt{G} = 0 \), and \( \lim_{\rho \to 0} (\sqrt{G})_{\rho} = 1 \),

\[ K(p) = -\lim_{\rho \to 0} \frac{\partial^3 \sqrt{G}}{\partial \rho^3} \]  
82.b

Now we can Taylor expand \( \sqrt{G} \) around the point \( p \),
\[ \sqrt{G}(\rho, \theta) = \sqrt{G}(0, \theta) + \rho (\sqrt{G})_{\rho}(0, \theta) + \frac{1}{2} \rho^2 (\sqrt{G})_{\rho \rho}(0, \theta) + \frac{1}{6} \rho^3 (\sqrt{G})_{\rho \rho \rho}(0, \theta) + \cdots \]  
83.

Substituting in our knowledge of \( \sqrt{G} \) and its derivatives at \( \rho = 0 \), we have,
\[ \sqrt{G}(\rho, \theta) = \rho - \frac{1}{6} \rho^3 K(\rho) + \cdots \]  
84.

We learn the first fundamental form in the vicinity of point \( p \),
\[ d\tilde{r}^2 = d\rho^2 + \rho^2 \left( 1 - \frac{1}{3} \rho^2 K(p) + \cdots \right) d\theta^2 \]  
85.

where we see that the curvature of the surface effects the distances along the geodesic circles, perpendicular to the radial geodesics emanating from point \( p \). Eq.85 is the generalization of the Eq. 16-19 that we calculated explicitly in spherical geometry.

Let’s use Eq. 85 to calculate the circumference \( C \) of a geodesic circle centered at the point \( p \), as discussed in the textbook,
\[ C(\rho) = \int_{0}^{2\pi} \sqrt{G}(\rho, \theta) d\theta = 2\pi \rho - \frac{\pi}{3} \rho^3 K(p) + \cdots \]  
86.a

Solving for \( K(p) \),
\[ K(p) = -\frac{3}{\pi} \lim_{\rho \to 0} \frac{2\pi \rho - C(\rho)}{\rho^3} \]  
86.b

which shows that the local violation of the Euclidean relation \( C(\rho) = 2\pi \rho \) tells us the curvature of the surface. If \( K > 0 \), then \( C(\rho) < 2\pi \rho \) and if \( K < 0 \), then \( C(\rho) > 2\pi \rho \).
We can also use this result to study how geodesics diverge from a given point. In general we are interested in the problem studied systematically by Jacobi: Consider a bundle of geodesics emanating from point $p$ as shown in Fig. 32,

![Fig. 32 A Jacobi field line perpendicular to radial geodesics](image)

We want to know how fast the geodesics diverge as $\rho$ grows. The “Jacobi Field” $\vec{J}(\rho, \theta)$, where $\theta$ is a coordinate perpendicular to the geodesics in the $\rho$ direction, is invented to capture this information. In fact Eq. 81 provides the answer to this general question in the context of Gaussian polar coordinates,

$$\frac{\partial^2}{\partial \rho^2} \sqrt{G} + K \sqrt{G} = 0$$

which is a simple form of the “Jacobi differential equation” which tells us how fast geodesics diverge or converge. This is clear if we consider the lengths of segments along geodesic circles as a function of $\rho$ for small $\rho$,
\[ L(\rho) = \int_{\theta_0}^{\theta_1} \sqrt{G(\rho, \theta)} \, d\theta \]

Inserting the Taylor expansion for \( \sqrt{g} \), we have,

\[ L(\rho) = \left( \rho - \frac{1}{6} \rho^3 K(p) + \cdots \right) \Delta \theta \]

So, if \( K(p) = 0 \), \( L(\rho) \) is linear. This is an obvious feature of planar Euclidean geometry. But if \( K(p) > 0 \), then \( L(\rho) \) increases slower, and if \( K(p) < 0 \), \( L(\rho) \) increases faster as shown in Fig. 33,

![Fig. 33 Geodesic deviation curves for vanishing, positive and negative Gaussian curvatures](image)

We made such a plot in the textbook on more intuitive grounds. The figure suggests in the case of surfaces with positive \( K \), that nearby geodesics may initially diverge as \( \rho \) grows, but then might approach on another. We saw that this was the case in Spherical geometry where we saw that geodesics emanating from a point \( p \) necessarily intersect at the point \( q \) anti-podal to \( p \). Using
Jacobi fields and the Gauss-Bonnet Theorem much more can be said and introductory remarks on these subjects will be made in the next section. These questions have important analogues in general relativity where the “geodesic deviation”, the space-time distance between nearby geodesics, captures a great deal of physics. For example, the analysis of the observability of gravitational waves uses these ideas as also discussed in the textbook.

Parallel Transport, Curvature, Holonomy and Jacobi Fields

In our previous discussion on Spherical geometry and the Gauss-Bonnet Theorem, we have seen that parallel transport and covariant differentiation have applications which provide insight into the local Gaussian curvature of a surface. We saw that the parallel transport of a vector around a geodesic triangle did not yield the same vector at the beginning and the end of its excursion, and the change of the vector was proportional to the surface’s local curvature. The path dependence of parallel transportation was seen to be related to the fact that the commutator of the different components of the covariant derivative is proportional to the local Gaussian curvature.

Now we should derive the generalization of Eq. 23-24 to a general curved surface. Consider

\[ \frac{D}{\partial v} \frac{D}{\partial u} \vec{r}_u \]. We begin with \( \frac{D}{\partial u} \vec{r}_u = \Gamma^1_{11} \vec{r}_u + \Gamma^2_{11} \vec{r}_v \) using the notation of Eq. 58 and where we observed that the normal components of \( \frac{D}{\partial v} \vec{r}_u \) in Eq. 58 do not contribute to the covariant derivative. Using Eq. 58 again, we find,

\[
\frac{D}{\partial v} \frac{D}{\partial u} \vec{r}_u = [(\Gamma^1_{11})_v + \Gamma^1_{12} \Gamma^1_{11} + \Gamma^2_{11} \Gamma^1_{12}] \vec{r}_u + [(\Gamma^2_{11})_v + \Gamma^2_{12} \Gamma^1_{11} + \Gamma^2_{12} \Gamma^2_{12}] \vec{r}_v
\]

Similarly,

\[
\frac{D}{\partial u} \frac{D}{\partial v} \vec{r}_u = [(\Gamma^1_{12})_u + \Gamma^1_{12} \Gamma^1_{11} + \Gamma^2_{12} \Gamma^1_{12}] \vec{r}_u + [(\Gamma^2_{12})_u + \Gamma^2_{11} \Gamma^1_{12} + \Gamma^2_{12} \Gamma^2_{12}] \vec{r}_v
\]

So,

\[
\left[ \frac{D}{\partial v}, \frac{D}{\partial u} \right] \vec{r}_u = [(\Gamma^1_{11})_v - (\Gamma^1_{12})_u + \Gamma^1_{22} \Gamma^1_{11} - \Gamma^1_{12} \Gamma^2_{12}] \vec{r}_u
\]

\[
+ [(\Gamma^2_{11})_v - (\Gamma^1_{12})_u + \Gamma^2_{12} \Gamma^1_{11} + \Gamma^2_{11} \Gamma^2_{22} - \Gamma^2_{11} \Gamma^1_{12} - \Gamma^2_{12} \Gamma^2_{12}] \vec{r}_v
\]
We have seen these particular collections of Christoffel symbols before! Using Eq. 63 and 64, we have

\[ \left[ \frac{D}{\partial v}, \frac{D}{\partial u} \right] \tilde{r}_u = K [E \tilde{r}_v - F \tilde{r}_u] = K [\tilde{r}_u \cdot \tilde{r}_u \tilde{r}_v - \tilde{r}_u \cdot \tilde{r}_v \tilde{r}_u] = K (\tilde{r}_u \times \tilde{r}_v) \times \tilde{r}_u \quad \text{88.a} \]

where we used the triple vector cross product identity \((\tilde{r}_u \times \tilde{r}_v) \times \tilde{r}_u = \tilde{r}_u \cdot \tilde{r}_u \tilde{r}_v - \tilde{r}_u \cdot \tilde{r}_v \tilde{r}_u\).

Note that Eq. 88.a can be written as,

\[ \left[ \frac{D}{\partial v}, \frac{D}{\partial u} \right] \tilde{r}_u = K |\tilde{r}_u \times \tilde{r}_v| R_{\pi/2}(\tilde{r}_u) \quad \text{88.b} \]

where \( R_{\pi/2}(\tilde{r}_u) \) means the vector resulting when \( \tilde{r}_u \) is rotated 90 degrees in the counterclockwise direction in the tangent plane. Eq. 88b is important because \( K |\tilde{r}_u \times \tilde{r}_v| \, du dv = K dA \), the product of the Gaussian curvature and the surface area element. We recall from the Gauss-Bonnet Theorem that it is this combination of geometrical factors that is particularly significant: \( \oint K dA \) determines the path dependence of parallel transport of vectors around closed, smooth, simple curves on the surface.

We can write Eq. 88a for the vector \( \tilde{r}_v \) by replacing \( u \to v \) and \( v \to u \) in it,

\[ \left[ \frac{D}{\partial u}, \frac{D}{\partial v} \right] \tilde{r}_v = K (\tilde{r}_u \times \tilde{r}_u) \times \tilde{r}_v \]

Or,

\[ \left[ \frac{D}{\partial u}, \frac{D}{\partial v} \right] \tilde{r}_v = K (\tilde{r}_u \times \tilde{r}_u) \times \tilde{r}_v \quad \text{88.c} \]

where we were careful with the ± signs and the order of vectors in the cross product, \( \tilde{r}_u \times \tilde{r}_v = -\tilde{r}_v \times \tilde{r}_u \).

We learn from results Eq. 88a-c that the commutator of the two components of the covariant derivative are proportional to the Gaussian curvature \( K \). The other ingredient in the formula consists of generic vector algebra.
This result confirms our suspicion that the Gaussian curvature \( K \) controls the path dependence of parallel transport. Since Eq.88a-c is a local identity it is the essence of our other results which considered finite translations along surfaces, etc.

Now let’s consider more useful generalizations of Eq. 88a-c. We begin with,

\[
\left[ \frac{D}{\partial v}, \frac{D}{\partial u} \right] \vec{V}(u, v)
\]

where \( \vec{V} \) is a general tangent vector field,

\[
\vec{V}(u, v) = a(u, v)\hat{r}_u + b(u, v)\hat{r}_v
\]

First we calculate,

\[
\frac{D}{\partial u} \vec{V}(u, v) = \frac{D}{\partial u} \left( a \hat{r}_u + b \hat{r}_v \right) = a \frac{D}{\partial u} \hat{r}_u + b \frac{D}{\partial u} \hat{r}_v + \frac{\partial a}{\partial u} \hat{r}_u + \frac{\partial b}{\partial u} \hat{r}_v
\]

Repeating this process,

\[
\frac{D}{\partial v} \frac{D}{\partial u} \vec{V}(u, v) = a \frac{D}{\partial v} \frac{D}{\partial u} \hat{r}_u + b \frac{D}{\partial v} \frac{D}{\partial u} \hat{r}_v + \frac{\partial a}{\partial v} \frac{D}{\partial u} \hat{r}_u + \frac{\partial b}{\partial v} \frac{D}{\partial u} \hat{r}_v + \frac{\partial^2 a}{\partial v \partial u} \hat{r}_u + \frac{\partial^2 b}{\partial v \partial u} \hat{r}_v
\]

The analogous result for \( \frac{D}{\partial u} \frac{D}{\partial v} \vec{V}(u, v) \) follows by interchanging variables. Then forming the commutator, we see that all the ordinary double derivatives cancel as expected and we are left with,

\[
\left[ \frac{D}{\partial v}, \frac{D}{\partial u} \right] \vec{V} = a \left[ \frac{D}{\partial v}, \frac{D}{\partial u} \right] \hat{r}_u + b \left[ \frac{D}{\partial v}, \frac{D}{\partial u} \right] \hat{r}_v
\]

Using Eq. 88a-c, we find,

\[
\left[ \frac{D}{\partial v}, \frac{D}{\partial u} \right] \vec{V} = K(\hat{r}_u \times \hat{r}_v) \times \vec{V}
\]

which provides a useful, elegant generalization for a generic tangent vector field \( \vec{V} \).

Using these results we can make some progress beyond our earlier discussion of Jacobi Fields, a systematic approach to understand the rate at which nearby geodesics converge or
diverge on a surface of Gaussian curvature $K(u, v)$. Our discussion will only apply to a region covered by a Gaussian polar coordinate mesh around a point $p$. More global and general discussions are left to the references [1-3]. To begin, write out Eq. 88a for Gaussian polar coordinates $u \rightarrow \rho$ and $v \rightarrow \theta$,

$$\left[ \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \rho} \right] \tilde{r}_\rho = K \left( \tilde{r}_\rho \times \tilde{r}_\theta \right) \times \tilde{r}_\rho$$ \hspace{1cm} (90).

But when $\theta = \theta_0$ is fixed $\tilde{r}(\rho, \theta)$ is a radial geodesic so $\frac{\partial}{\partial \rho} \tilde{r}_\rho = 0$. Furthermore,

$$\frac{\partial}{\partial \theta} \tilde{r}_\rho = \frac{\partial}{\partial \theta} \frac{\partial}{\partial \rho} = \left( \tilde{r}_\rho \theta \right)_T = \left( \tilde{r}_\rho \theta \right)_T = \frac{\partial}{\partial \rho} \tilde{r}_\theta \hspace{1cm} (91).$$

where the subscript “T” means “projection onto the tangent plane”. Therefore,

$$\left[ \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \rho} \right] \tilde{r}_\rho = \frac{\partial}{\partial \theta} \frac{\partial}{\partial \rho} \tilde{r}_\rho - \frac{\partial}{\partial \rho} \frac{\partial}{\partial \theta} \tilde{r}_\rho = - \frac{\partial}{\partial \rho} \frac{\partial}{\partial \theta} \tilde{r}_\theta \hspace{1cm} (92).$$

Now Eq. 90 becomes,

$$\frac{\partial}{\partial \rho} \frac{\partial}{\partial \theta} \tilde{r}_\rho + K (\tilde{r}_\rho \times \tilde{r}_\theta) \times \tilde{r}_\rho = 0 \hspace{1cm} (93.a)$$

This is an example of a Jacobi equation. Since $\rho$ is an arclength, call it s. Denote $\tilde{r}_\theta (s, \theta) = \tilde{f}(s)$, a Jacobi field. Then Eq. 93a reads,

$$\frac{\partial}{\partial s} \frac{\partial}{\partial \theta} \tilde{f}(s) + K \left( \tilde{r}_\rho \times \tilde{f}(s) \right) \times \tilde{r}_s = 0 \hspace{1cm} (93.b)$$

We recognize that this is the generalization of Eq. 15, originally introduced in the context of Spherical geometry. Let’s continue that discussion here. We consider a unit sphere and locate points on it with spherical polar coordinates, $\tilde{r}(\theta, \varphi)$. Consider the equator $\theta = \pi/2$ and the segment along the equator’s great circle, $0 < \varphi < \pi$, which goes half way around the sphere, $\tilde{r}(\theta, \varphi) = \tilde{r}(\pi/2, \varphi)$. Consider the direction $\tilde{u}(\varphi)$ with the initial condition $\tilde{u}(\varphi = 0) \cdot \tilde{i}_\varphi (\varphi = 0) = 0$ and let $\tilde{u}(\varphi)$ be the parallel transport of $\tilde{u}(\varphi = 0)$ to the range $0 < \varphi < \pi$. Then it is easy to see that $\tilde{u}(\varphi) \sin \varphi$ satisfies the Jacobi equation. Calculate

$$\frac{D \tilde{f}}{\partial \varphi} = (\cos \varphi) \tilde{u}(\varphi) \hspace{1cm} , \hspace{1cm} \frac{D}{\partial \varphi} \frac{D \tilde{f}}{\partial \varphi} = (- \sin \varphi) \tilde{u}(\varphi)$$
where we used $D\bar{u}(\varphi)/\partial \varphi = 0$ because $\bar{u}(\varphi)$ is defined through parallel transport. Eq. 91b becomes,

$$\frac{D}{\partial \varphi} \frac{D}{\partial \varphi} \vec{J} + K(\vec{r}_\varphi \times \vec{J}) \times \vec{r}_\varphi = - (\sin \varphi) \bar{u}(\varphi) + (\sin \varphi) \bar{u}(\varphi) = 0$$

Here we observed that $\vec{r}_\varphi(\varphi) \cdot \vec{J}(\varphi) = 0$ for $0 < \varphi < \pi$ and $\vec{r}_\varphi(\varphi) \times (\vec{r}_\varphi(\varphi) \times \bar{u}(\varphi)) \times \vec{r}_\varphi(\varphi) = \bar{u}(\varphi)$. We note that $\vec{J}(\varphi)$ vanishes at $\varphi = 0$ and $\varphi = \pi$ and has harmonic dependence on $\varphi$ in between. Points where Jacobi fields vanish are called conjugate points and on the sphere they are points $p$ and their anti-podal images $q$. Clearly this simple behavior is special to the sphere: on other compact surfaces the conjugate image of a point $p$ is typically a curve which is swept out as the angle between the two geodesics at point $p$ varies. Another interesting fact is that on surfaces with $K \leq 0$ the second intersection point $q$ of two geodesics that intersect at point $p$ goes to infinity as the angle between these geodesics at the point $p$ goes to zero. This behavior is rather subtle. Take the cylinder as an example. $K = 0$ here. Consider a meridian which is clearly a geodesic. (We identify all the geodesics of the cylinder by “unrolling” it and drawing straight lines on the resulting flat surface (which has identified edges).) Clearly other geodesics inclined at a finite angle intersect the meridian an infinite number of times. The global topology of the surface is critical here.

When we study “geodesic deviation” in general relativity, we will derive the four dimensional analog of Eq. 93b. The Riemann curvature tensor will play the role of the Gaussian curvature $K$ there. Since curvatures are geometric, physical quantities, independent of the coordinate mesh used to parametrize the manifold in which they exist, this equation is particularly useful.

References

4. P. M. H. Wilson, Curved Spaces, From Classical Geometries to Elementary Differential