Supplemental Lecture 5

Thomas Precession and Fermi-Walker Transport in Special Relativity and Geodesic Precession in General Relativity

Abstract

This lecture considers two topics in the motion of tops, spins and gyroscopes: 1. Thomas Precession and Fermi-Walker Transport in Special Relativity, and 2. Geodesic Precession in General Relativity. A gyroscope (“spin”) attached to an accelerating particle traveling in Minkowski space-time is seen to precess even in a torque-free environment. The angular rate of the precession is given by the famous Thomas precession frequency. We introduce the notion of Fermi-Walker transport to discuss this phenomenon in a systematic fashion and apply it to a particle propagating on a circular orbit. In a separate discussion we consider the geodesic motion of a particle with spin in a curved space-time. The spin necessarily precesses as a consequence of the space-time curvature. We illustrate the phenomenon for a circular orbit in a Schwarzschild metric. Accurate experimental measurements of this effect have been accomplished using earth satellites carrying gyroscopes.

This lecture supplements material in the textbook: Special Relativity, Electrodynamics and General Relativity: From Newton to Einstein (ISBN: 978-0-12-813720-8). The term “textbook” in these Supplemental Lectures will refer to that work.

Keywords: Fermi-Walker transport, Thomas precession, Geodesic precession, Schwarzschild metric, Gravity Probe B (GP-B).

Introduction

In this supplementary lecture we discuss two instances of precession: 1. Thomas precession in special relativity and 2. Geodesic precession in general relativity.
To set the stage, let’s recall a few things about precession in classical mechanics, Newton’s world, as we say in the textbook.

The precession of a “spin” vector $\hat{s}$ at angular velocity $\omega$ about a fixed axis is described by the first order differential equation,

$$\frac{d\hat{s}}{dt} = \vec{\omega} \times \hat{s}$$  \hspace{1cm} (1)

If $\vec{\omega}$ points in the z direction, $\vec{\omega} = \omega \hat{k}$, then $\hat{s}$ rotates around it at angular velocity $\omega$. The quantities $\hat{s} \cdot \hat{s}$ and $s_z$ are conserved. To see this, “dot” $\hat{s}$ into Eq. 1,

$$\hat{s} \cdot \frac{d\hat{s}}{dt} = \frac{1}{2} d(\hat{s} \cdot \hat{s}) = \hat{s} \cdot (\vec{\omega} \times \hat{s}) = 0$$  \hspace{1cm} (2a)

and if we dot $\hat{k}$ into Eq. 1,

$$\hat{k} \cdot \frac{d\hat{s}}{dt} = \frac{ds_z}{dt} = \hat{k} \cdot (\vec{\omega} \times \hat{s}) = 0$$  \hspace{1cm} (2b)

which proves the two points. Finally, to find $s_x(t)$ and $s_y(t)$ write out Eq.1 in Cartesian coordinates and use matrix notation,

$$\frac{d}{dt} \begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix} = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix}$$  \hspace{1cm} (3)

Note that the matrix in Eq. 2 is antisymmetric (\(\sum_{jk} \epsilon_{ijk} \omega^j \hat{s}^k\)) and this guarantees that time evolution preserves the norm of the spin (this point was also made in Eq. 2a). The two coupled equations in Eq. 3 read,

$$\frac{ds_x}{dt} = -\omega s_y \quad \frac{ds_y}{dt} = +\omega s_x$$

To solve this coupled set of differential equations, differentiate the first one, substitute into the second one and find a harmonic oscillator equation for $s_x$. Similarly for $s_y$. Then for the initial conditions \( \left(s_x(0), s_y(0), s_z(0)\right) = (s, 0, 0) \), the solution is \( \left(s_x(t), s_y(t), s_z(t)\right) = (s \cos \omega t, s \sin \omega t, 0) \). The spin precesses as shown in Fig. 1.
When we turn to Thomas precession [1] we will consider the spin of a particle on a curve in Minkowski space. The particle will experience a four velocity $u^\mu(t)$ and four acceleration $a^\mu(t)$ on the curve and the spin $S^\mu(t) = (s^0(t), \vec{s}(t))$ will be seen to precess in the lab frame,

$$\frac{d\vec{s}}{dt} = \vec{\omega}_T \times \vec{s}$$  \hspace{1cm} (4a)

where $\vec{\omega}_T$ is the Thomas angular velocity [1],

$$\vec{\omega}_T = \frac{\gamma^2}{\gamma+1} \frac{\vec{a} \times \vec{v}}{c^2}$$  \hspace{1cm} (4b)

Here $\vec{v}$ is the particle’s velocity in the lab frame and $\vec{a}$ is its acceleration. Eq 4a and 4b are exact in special relativity. We see that $\omega_T$ is different from zero if the acceleration “curves” the particle’s trajectory. Recall from the textbook discussion of Thomas precession that a calculation of the precession of the spin over a full cycle of its closed simple path was possible just knowing Lorentz contraction [2,3]. The result in Eq. 4 goes much further and shows how it develops differentially. This equation has wider and more general applicability.
Now consider precession in general relativity. There are many examples of precession in general relativity but we will consider only the simplest: the precession of the spin of a massive particle as it moves freely outside a fixed mass, described by the Schwarzschild metric derived in Chapter 12 of the textbook. We already know to expect precession due to the curvature $R^\mu_{\alpha\beta\gamma}$ of the space: parallel transport of a vector around an infinitesimal closed path produces a non-zero result when $R^\mu_{\alpha\beta\gamma} \neq 0$ inside the path. Here we will be interested in the precession of a gyroscope on a satellite orbiting the earth as it undergoes many polar orbits over a long period of time [4]. The general relativistic predictions for such an experiment were well confirmed by the Gravity Probe B (GP-B) launched and operated by NASA from 2004-2011 [5]. In addition to confirming geodesic precession the probe also measured “frame-dragging” [6] a phenomenon that goes beyond the Schwarzschild metric and accounts for the spinning of the earth on its axis. The rotation is described by the Kerr metric which contains gravito-magnetic effects that produce additional precession effects (Lense-Thirring precession) which is approximately one-hundredth the magnitude of the effect studied here. This phenomenon will be the subject of a later supplementary lecture.

Fermi-Walker Transport and Thomas Precession

We seek the special relativistic generalization of precession. We are particularly interested in the first order equation of motion of the spin of a massive particle as it travels on a curve and experiences acceleration. This is the situation in the study of relativistic spectroscopy of atoms where the electrons experience the electromagnetic field of the nucleus.

We begin with a more general situation. Consider a four vector $A^\mu$ attached to the particle which is traveling on a curve in the lab inertial frame. Its time evolution is expected to be given by a first order differential equation of the covariant form [4],

$$\frac{dA^\mu}{ds} = \sum_\alpha K^\mu_{\alpha} A^\alpha$$

where $ds$ is the invariant length variable. It will become the differential of proper time, $c d\tau$, in the applications that follow. Here $K^{\mu\rho}$ is a second rank tensor which must be constructed out of variables that describe the particle: the particle’s four velocity $u^\mu$ and acceleration $a^\mu$. The
Tensor must also be anti-symmetric, so that it preserves the length of $A^\alpha$, as illustrated for Newtonian mechanics in the Introduction. Let's apply Eq. 5 to the time evolution of a four vector we understand thoroughly, $u^\mu$,

$$\frac{du^\mu}{d\tau} = a^\mu = c \sum_\alpha K^\mu_\alpha u^\alpha$$

This equation works if we choose,

$$c K^{\mu \rho} = \frac{1}{c^2} (a^\mu u^\rho - u^\mu a^\rho)$$

In this case,

$$c \sum_\alpha K^\mu_\alpha u^\alpha = \frac{1}{c^2} \sum_\alpha (a^\mu u_\alpha - u^\mu a_\alpha) u^\alpha = a^\mu$$

where we used $\sum_\alpha a_\alpha u^\alpha = 0$ and $\sum_\alpha u_\alpha u^\alpha = c^2$. (The result $\sum_\alpha a_\alpha u^\alpha = 0$ was discussed in the textbook and follows from the fact that in the particle’s instantaneous rest frame, the spatial components of the four velocity $u^\mu$ vanish and the temporal component of the four acceleration $a^\mu$ vanishes.)

We arrive at the Fermi-Walker transport equation [4],

$$\frac{dA^\mu}{d\tau} = \frac{1}{c^2} \sum_\alpha (a^\mu u_\alpha - u^\mu a_\alpha) A^\alpha$$

(6)

Our task is to apply Eq. 6 to the time evolution of the spin of the traveling particle. Denote the spin four vector $S^\mu$.

In order to compare to the results on Thomas precession obtained in the text, we place the particle on a circular trajectory [4],

$$x = r \cos \omega \tau, \quad y = r \sin \omega \tau, \quad z = 0$$

The four velocity components, $dx^\mu / d\tau = u^\mu$, are then,

$$u^1 = -\omega r \sin \omega \tau \quad u^2 = \omega r \cos \omega \tau \quad u^3 = 0$$
The value of $u^0$ follows from the fact that the four vector has norm $\sum_\mu u^\mu u_\mu = c^2 = (u^0)^2 - (u^1)^2 - (u^2)^2$ which implies that $u^0 = c\sqrt{1 + (\omega \tau / c)^2} = c\gamma$.

We can also write the phases of the orbital motion, $\omega \tau$, in term of $t$, the time variable in the inertial lab frame. Note that $dt / d\tau = \gamma$, so $t = \gamma \tau$ and $\omega \tau = \Omega t$, where $\Omega = \omega / \gamma$.

The Fermi-Walker transport equation for the spin four vector reads,

$$\frac{dS^\mu}{d\tau} = \frac{1}{c^2} \sum_\alpha (a^\mu u_\alpha - u^\mu a_\alpha) S^\alpha$$

which simplifies because $\sum_\mu u_\mu S^\mu = 0$ (In the particle’s rest frame $u'^\mu = (c, 0, 0, 0)$ and $S'^\mu = (0, \vec{s})$, so $\sum_\mu u'_\mu S'^\mu = 0$. Since the inner product is Lorentz invariant, the result must hold in all frames.) Now we have,

$$\frac{dS^\mu}{d\tau} = -\frac{1}{c^2} \sum_\alpha (u^\mu \frac{du_\alpha}{d\tau}) S^\alpha$$  \hspace{1cm} (7)

Since $u^3 = 0$, we learn that $dS^3 / d\tau = 0$ so $dS^3 / dt = 0$. So if we take $S^3 = 0$ initially then it remains zero for later $t$. The value of $S^0$ follows from the fact that $S \cdot u = 0$, $0 = u^0 S^0 - u^1 S^1 - u^2 S^2$, so $S^0 = (u^1 S^1 + u^2 S^2) / u^0$. So, if we can find $S^1(t)$ and $S^2(t)$, then we are done.

Now we can write Eq. 7 for $S^1(t)$ and $S^2(t)$,

$$\frac{dS^1}{d\tau} = -\frac{1}{c^2} \sum_\alpha (u^1 \frac{du_\alpha}{d\tau}) S^\alpha = \left( -\frac{\omega \tau \sin \omega \tau}{c^2} \right) \left( \frac{du_1}{d\tau} S^1 + \frac{du_2}{d\tau} S^2 \right) \hspace{1cm} (8)$$

where the term $du_0 / d\tau = 0$ because $u_0 = \sqrt{1 + (\omega \tau / c)^2}$ is a constant. We can work on the right hand side of Eq. 8,

$$\frac{du_1}{d\tau} S^1 + \frac{du_2}{d\tau} S^2 = \omega^2 r \cos \omega \tau S^1 + \omega^2 r \sin \omega \tau S^2$$

Substituting into Eq. 8 and doing similar manipulations for $dS^2 / d\tau$ we arrive at a matrix differential equation with sinusoidal coefficients,

$$\frac{d}{dt} \begin{pmatrix} S^1 \\ S^2 \end{pmatrix} = (\gamma^2 - 1) \Omega \begin{pmatrix} \sin \Omega t \cos \Omega t & \sin^2 \Omega t \\ -\cos^2 \Omega t & -\sin \Omega t \cos \Omega t \end{pmatrix} \begin{pmatrix} S^1 \\ S^2 \end{pmatrix} \hspace{1cm} (9)$$
where we used $\omega^2 r^2/c^2 = \gamma^2 - 1$, $dt/d\tau = \gamma$ and $\omega = \Omega t$ to simplify the prefactor and the arguments of the sinusoidal functions.

One way to solve a differential equation with sinusoidal coefficients is with Fourier transforms. In the appendix to this lecture we solve it by trial and error. For the initial conditions $S^1(0) = s$ and $S^2(0) = 0$, the solution is [4],

\begin{align*}
S^1(t) &= \frac{1}{2} s [(1 + \gamma) \cos(1 - \gamma)\Omega t + (1 - \gamma) \cos(1 + \gamma)\Omega t] \\
S^2(t) &= \frac{1}{2} s [(1 + \gamma) \sin(1 - \gamma)\Omega t + (1 - \gamma) \sin(1 + \gamma)\Omega t]
\end{align*} \tag{10}

Eq. 10 has some familiar features. We read off the Thomas angular frequency from the first terms in Eq. 10,

$$
\Omega_T = (\gamma - 1) \Omega
$$ \tag{11}

This is the result for Thomas precession found in the textbook using Lorentz contraction [2,3]. For non-relativistic motion, $\gamma - 1 \approx v^2/2c^2 \ll 1$ and we have the expression used in calculations of relativistic corrections of atomic spectroscopy. A plot of the “large” first terms in Eq. 10, those proportional in amplitude to $(1 + \gamma)$, shows that the spin precesses in the opposite direction to the motion of the particle around its orbit. This point was also made in the cryptic argument in the textbook. Another feature of the solution Eq. 10 is that it consists of a “large” amplitude “slow” term and a “small” amplitude “fast” term in the non-relativistic limit. It is not a simple single frequency rotation. This behavior is typical of gyroscopic precession [7]: as the spin rotates in the clockwise direction with the Thomas angular frequency, it simultaneously executes a small amplitude rapid rotation in the opposite direction.

The student is encouraged to master reference [8], which is “linked” below in the references, for a derivation of the Thomas precession equation, Eq. 4. This is the most useful and fundamental result. The derivation is at the level of the textbook and shows clearly that the Thomas precession occurs because the curvature in the particle’s flight, as measured in the lab inertial frame, makes the Lorentz boost from the lab frame to the rest frame of the particle time dependent. This is the subtle effect which produces a non-zero $\Omega_T$. The treatment in [8] obtains the equation of motion of the spin in its instantaneous rest frame. There the motion is a pure precession with a single frequency, the Thomas value. The approach presented here considers the
spin four vector in the lab frame. Its equation of motion is more complicated, Eq. 9 is not anti-symmetric so it is not a pure rotation, as we saw in the solution Eq. 10.

Geodesic Precession in General Relativity

We are interested in the motion of a particle of a non-zero mass around a large static mass $M$ [4]. The metric describing space-time outside of $M$ is the Schwarzschild metric,

$$ds^2 = (1 - r_s/r)c^2 dt^2 - (1 - r_s/r)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(12)

where $r_s$ is the Schwarzschild radius $r_s = 2GM/c^2$. In the textbook we determined the conserved quantities, the energy $e$ and the angular momentum $h$,

$$(1 - r_s/r) \frac{dt}{d\tau} = e, \quad r^2 \dot{\phi} = h$$

(13)

Then in Chapter 12 we obtained the first integral of the equation of motion (the geodesic equation),

$$\dot{r}^2 = e^2 - V_{eff}(r), \quad V_{eff} = (1 - r_s/r) \left( \frac{h^2}{r^2} + 1 \right)$$

(14)

$V_{eff}$ has the shape shown in Fig. 2 The “dot” in Eq. 14 denotes differentiation with respect to the proper time $\tau$. The shape of $V_{eff}$ follows from the following effects: For large $r$, $V_{eff}$ is normalized to unity, as $r$ decreases Newton’s $1/r$ potential pulls $V_{eff}$ down, then as $r$ decreases more the centrifugal barrier pushes $V_{eff}$ up and finally the attractive non-linear relativistic $1/r^3$ term pulls $V_{eff}$ to minus infinity.
We are particularly interested in circular orbits where $r^2 = 0$. The expression for $V_{\text{eff}}(r)$ shows that $r = r_s \frac{h^2}{r_s^2} \left(1 + \sqrt{1 - 3r_s^2/H^2}\right)$ is a local minimum of $V_{\text{eff}}$, $dV_{\text{eff}}/dr = 0$, $d^2V_{\text{eff}}/dr^2 > 0$, as shown in the figure. (In this case the minimum, labeled $r$ in the figure, occurs near 4.31. The minimum is quite flat in this example.) Then $h$ and $e$ can be solved in terms of $r$. Algebra yields,

$$h = \sqrt{\frac{r_s^2/2}{(1-3r_s/2r)}}, \quad e = \frac{(1-r_s/r)}{(1-3r_s/2r)}$$

(15)

Our last preliminary is to find the period of the circular orbit in terms of its radius $r$. The conservation law $r^2 \dot{\phi} = h$ implies that $d \tau = \frac{r^2}{ch} d\phi$ so for a full revolution, $\Delta \phi = 2\pi$, $\Delta \tau = 2\pi r^2/\text{ch}$. Eq. 15 then gives,

$$\Delta \tau = \frac{2\pi \tau^2}{\text{ch}} = \frac{2\pi r}{c} \sqrt{\frac{2r}{r_s^2} \left(1 - \frac{3r_s}{2r}\right)}$$

(16)

The first conservation law in Eq. 13, $(1 - r_s/r) \frac{dt}{d\tau} = e$, implies, using Eq. 16,

$$\Delta t = \frac{e}{(1-r_s/r)} \Delta \tau = \frac{2\pi r}{c} \sqrt{\frac{2r}{r_s^2}}$$

(17)
With these preliminaries done, we are ready to consider geodesic precession. Suppose that the mass in orbit at radius \( r \) carries a gyroscope or classical spin. There are no torques on the spin, so it moves on a geodesic, and it parallel transports and satisfies,

\[
\frac{d}{d\tau} \mathbf{s}^\mu + c \sum_{\alpha\beta} \Gamma^\mu_{\alpha\beta} \mathbf{u}^\alpha \mathbf{s}^\beta = 0
\]

The student should review Chapter 12 of the textbook for the discussion of geodesic propagation in a curved space-time. In this application \( \mathbf{u}^\mu \) is the four velocity of the mass moving freely in a space-time described by the Schwarzschild metric. We suppose that the particle moves on a circular orbit at fixed \( \theta = \pi/2 \), and using the notation where a dot indicates \( d/d\tau \), for the Schwarzschild metric,

\[
\mathbf{u}^\mu = (c\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}) = (c\dot{t}, 0, 0, \dot{\phi}) = \left( (1 - \frac{3r_s}{2r})^{-1/2}, 0, 0, \frac{1}{r^2} \sqrt{\frac{r r_s/2}{(1-3r_s/2r)}} \right)
\]

where we used Eq. 15.

Finally we need the Christoffel symbols. These were computed in Chapter 12 of the textbook and read,

\[
\Gamma^1_{00} = \frac{r_s}{2r^2} (1 - r_s/r), \quad \Gamma^1_{33} = -r(1 - r_s/r), \quad \Gamma^2_{12} = 1/r, \quad \Gamma^3_{13} = 1/r
\]

with other components of the Christoffel symbols vanishing.

We have additional constraints on the spin four vector. First, \( \Sigma s^\mu s_\mu = -1 \), which reads, for \( \theta = \pi/2 \),

\[
g_{00}(s^0)^2 + g_{11}(s^1)^2 + g_{22}(s^2)^2 + g_{33}(s^3)^2 = -1
\]

so,

\[
(1 - r_s/r)(s^0)^2 - (1 - r_s/r)^{-1}(s^1)^2 - r^2(s^2)^2 - r^2(s^3)^2 = -1
\]

In addition there is the invariant, \( \Sigma s^\mu s_\mu = 0 \), which follows from the fact that \( u \) is purely time-like in the particle’s rest frame and \( s \) is purely space-like there. Writing out this expression \( \Sigma u^\mu s_\mu = 0 \),
(1 - r_s/r)(1 - 3r_s/2r)^{-1/2}s^0 - r^2 \frac{1}{r^2} \sqrt{\frac{rr_s/2}{(1-3r_s/2r)}} s^3 = 0

So, in the lab frame,

\[ s^0 = \frac{\sqrt{rr_s/2}}{(1-r_s/r)} s^3 \]  \hspace{1cm} (20)

With these preliminaries we can focus on the geodesic equations for the spatial components of the spin,

\[ \frac{d}{d\tau} s^1 + c \sum_{\alpha\beta} \Gamma^1_{\alpha\beta} u^\alpha s^\beta = \frac{d}{d\tau} s^1 + c \Gamma^1_{00} u^0 s^0 + c \Gamma^1_{33} u^3 s^3 = 0 \]

Now substitute Eq. 18, 19 and 20 into this expression and do some algebra to find,

\[ \frac{d}{d\tau} s^1 = \frac{c}{r} \sqrt{\frac{rr_s}{2}} \sqrt{1 - 3r_s/2r} s^3 \] \hspace{1cm} (21a)

Next the geodesic equation for \( s^2 \) is particularly simple because \( u^1 = u^2 = 0 \) eliminates the non-trivial terms,

\[ \frac{d}{d\tau} s^2 = 0 \] \hspace{1cm} (21b)

Finally, the geodesic equation for \( s^3 \) is,

\[ \frac{d}{d\tau} s^3 + c \sum_{\alpha\beta} \Gamma^3_{\alpha\beta} u^\alpha s^\beta = \frac{d}{d\tau} s^1 + c \Gamma^3_{31} u^3 s^1 = 0 \]

Using Eq. 18 and 19 produces,

\[ \frac{d}{d\tau} s^3 = \frac{c}{r^2} \sqrt{\frac{rr_s/2}{1-3r_s/2r}} s^1 \] \hspace{1cm} (21c)

So, we end up with three first order differential equations with constant (time independent) coefficients. If we differentiate Eq. 21c and substitute Eq. 21a into the result, we find that \( s^3 \) satisfies a harmonic oscillator equation,

\[ s^3 + \frac{c^2 r_s}{2r^3} s^3 = 0 \] \hspace{1cm} (22)
where the double dot is short hand for $d^2/d\tau^2$. If initially $s^3(0) = 0$, then

$$s^3(\tau) = s \sin \omega \tau$$

where

$$\omega = c \sqrt{\frac{r_s}{2r^3}} = \sqrt{\frac{GM}{r^3}}$$

and we used the value for the Schwarzschild radius, $r_s = 2GM/c^2$.

We need to compare the frequency of the spin precession to the frequency of the orbital motion. The orbital frequency follows from Eq. 16 for the period of the orbital motion,

$$\omega_{orb} = \frac{2\pi}{\Delta \tau} = c \sqrt{\frac{r_s}{2r^3}} (1 - 3r_s/2r)^{-1/2} = \omega(1 - 3r_s/2r)^{-1/2}$$

We learn that the rotation of the spin of the particle lags slightly behind the orbital rotation. The precession angle per revolution is then,

$$2\pi - \omega \Delta \tau = 2\pi (1 - \sqrt{1 - 3r_s/2r}) \approx 3\pi r_s/2r$$

where we assumed $r_s/r \ll 1$ in the final inequality.

Eq. 25 has been tested with exquisite accuracy by the satellite Gravity Probe B (GP-B) which was operated by NASA from 2004 to 2011 [5]. This space mission launched a satellite containing four gyroscopes in low earth orbit and accumulated data on their precession relative to a distant star, IM Pegas. The experiment measured geodesic precession to approximately 1%. It also measured another precession effect, the Lense-Thirring effect [6], due to the rotation of the earth to approximately 10%. This effect will be discussed in another Supplementary lecture in the future.

Geodesic precession has also been measured in strongly curved space-time through the observation of the spins of binary pulsars [9].

Appendix
Let’s solve the differential equation Eq. 9 which describes Thomas precession in special relativity,

\[
\frac{d}{dt} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = (\gamma^2 - 1)\Omega \begin{pmatrix} \sin \Omega t \cos \Omega t & \sin^2 \Omega t \\ -\cos^2 \Omega t & -\sin \Omega t \cos \Omega t \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}
\]

with the initial conditions \( S_1(0) = a \) and \( S_2(0) = 0 \).

Let’s proceed by “trial and error” using some observations about precession made in the introductory comments to the lecture.

We try a solution that describes simple precession.

\[
S_1 = a \cos \alpha t, \quad S_2 = a \sin \alpha t \quad (A.1)
\]

where the parameters \( a \) and \( \alpha \) should be determined from the differential equation. The first component of Eq. 9 reads,

\[
\frac{d}{dt} S_1 = (\gamma^2 - 1)\Omega (\sin \Omega t \cos \Omega t \ S_1 + \sin^2 \Omega t \ S_2)
\]

Substituting the guess Eq. A.1 into this expression,

\[
-a a \sin \alpha t = (\gamma^2 - 1) \Omega a \sin \Omega t (\cos \Omega t \cos \alpha t + \sin \Omega t \sin \alpha t)
\]

\[
-a a \sin \alpha t = (\gamma^2 - 1) \Omega a \sin \Omega t (\cos(\Omega - \alpha) t)
\]

where we used the trig identity \( \cos A \cos B + \sin A \sin B = \cos(A - B) \). Next use the trig identity \( 2 \sin A \cos B = \sin(A - B) + \sin(A + B) \), so the differential equation becomes,

\[
-a \sin \alpha t = \frac{1}{2} (1 - \gamma^2) \Omega [\sin \alpha t + \sin(2\Omega - \alpha) t]
\]

We learn that \( \alpha = \Omega \) in order to match the time dependences on both sides of the equation, but then the coefficients do not match and the trial solution fails!

We learn that the trial solution was too simple to accommodate both the frequencies and amplitude of the differential equation Eq. 9. Let’s try a slightly more general trial function: the sum of two precession terms so we can accommodate those two aspects of the differential equation,
\[ S_1 = a \cos at + b \cos \beta t , \quad S_2 = a \sin at + b \sin \beta t \]

Now the left hand side of the first component of the differential equation reads,

\[
aa \sin at + b \beta \sin \beta t = \frac{(aa + b\beta)}{2} (\sin at + \sin \beta t) + \frac{(aa - b\beta)}{2} (\sin at - \sin \beta t) \\
= (aa + b\beta) \sin \frac{(\alpha + \beta)t}{2} \cos \frac{(\alpha - \beta)t}{2} + (aa - b\beta) \sin \frac{(\alpha - \beta)t}{2} \cos \frac{(\alpha + \beta)t}{2} \tag{L}
\]

where we used the trig identities, \( \sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \) and \( \sin A - \sin B = 2 \sin \frac{A-B}{2} \cos \frac{A+B}{2} \).

On the right hand side,

\[
(1 - \gamma^2) \Omega [a \sin \Omega t \cos \Omega t \cos at + b \sin \Omega t \cos \Omega t \cos \beta t + a \sin^2 \Omega t \sin at + b \sin^2 \Omega t \sin \beta t ]
\]

The quantity in brackets can be rewritten in the same form as the left side,

\[
[a \sin \Omega t (\cos \Omega t \cos at + \sin \Omega t \sin at) + a \sin \Omega t (\cos \Omega t \cos \beta t + \sin \Omega t \sin \beta t)]
\]

Using the trig identity \( \cos A \cos B + \sin A \sin B = \cos(A - B) \) twice, this expression becomes,

\[
[a \sin \Omega t \cos(\Omega - \alpha) t + a \sin \Omega \cos(\Omega - \beta) t] \tag{R}
\]

Now compare the left (L) and right (R) hand sides. Their time dependences match if

\[
\frac{\alpha + \beta}{2} = \Omega , \quad a\alpha = b\beta \tag{A.2}
\]

The magnitudes of the two sides match if,

\[
2a^2 = (1 - \gamma^2) \Omega (\alpha + b)
\]

The right hand side of this expression can be simplified using Eq. A.2,

\[
2a^2 = (1 - \gamma^2) \Omega (\alpha + b) = (1 - \gamma^2) \Omega a (1 + b/a) = (1 - \gamma^2) \Omega a (1 + a/\beta)
\]

Finally, canceling a on both sides, multiplying through by \( \beta \) and using Eq. A.2 again,

\[
2\alpha \beta = (1 - \gamma^2) \Omega (\alpha + \beta) = 2(1 - \gamma^2) \Omega^2 \tag{A.3}
\]

which reduces to \( \alpha \beta = (1 - \gamma)(1 + \gamma) \Omega^2 \).
So, our final task is to solve Eq. A.2 and A.3,

\[ \alpha \beta = (1 - \gamma)(1 + \gamma)\Omega^2 \quad , \quad \alpha + \beta = 2 \Omega \]  \hspace{1cm} (A.4)

The resulting quadratic equation gives,

\[ \alpha = (1 - \gamma) \Omega \quad , \quad \beta = (1 + \gamma) \Omega \]  \hspace{1cm} (A.5)

Finally, the parameters \(a\) and \(b\) follow from Eq. A.2,

\[ \frac{b}{a} = \frac{\alpha}{\beta} = \frac{(1-\gamma)}{(1+\gamma)} \]  \hspace{1cm} (A.6)

Since the overall scale of the spin is not determined by the precession equation, we can choose,

\[ a = (1 + \gamma) \quad , \quad b = (1 - \gamma) \]  \hspace{1cm} (A.7)

Collecting everything, the solution for the spin reads,

\[ S^1 = \frac{1}{2} s [(1 + \gamma) \cos(1 - \gamma) \Omega \, t + (1 - \gamma) \cos(1 + \gamma) \Omega \, t] \]

\[ S^2 = \frac{1}{2} s [(1 + \gamma) \sin(1 - \gamma) \Omega \, t + (1 - \gamma) \sin(1 + \gamma) \Omega \, t] \]

as stated in the text.

One can check that the second differential equation for \(S^2(t)\), Eq. 9, is also satisfied by this result.

References


http://einstein.stanford.edu/


